Problem set 5, due in Wednesday 19/11/09

In this problem, we shall derive the expression for the Silk damping timescale using the 2-fluid formalism outlined in my lecture and in chapter 11.8. Please proceed as follows:

a. Begin with equations (11.8.1) and (11.8.2). Notice there is an error in (11.8.2b); it should be not $\tau_{e\gamma}$ but ... .

b. Set expansion of the universe $\dot{a}$ and gravity G to zero. Assume that the dynamical quantities vary as $e^{-i\omega t}$. Show that the equations reduce to the following:

$$\begin{aligned}(\omega^2 - k^2 v_{\rm sm}^2) v_m &= i\omega\delta v/\tau_{e\gamma} \\ (\omega^2 - k^2 v_{\rm sr}^2) v_r &= -i\omega\delta v/\tau_{\gamma e}\end{aligned}$$

where $\delta v = v_r - v_m$.

c. Show that this equations have the same mathematical form as those describing a pair of harmonic oscillators with angular frequencies $\omega_m = kv_{\rm sm}$ and $\omega_r = kv_{\rm sr}$ which are coupled by viscous friction. What do you expect to happen if the friction coefficient becomes very large? (Nice if you can do this but not really essencial for the rest of the problem).

d. From equations in (b), derive the following dispersion relation (hint: rewrite in a matrix form, certain determinant has to be zero):

$$(\omega^2 - \omega_m^2)(\omega^2 - \omega_r^2) + \frac{i\omega}{\tau_{e\gamma}\tau_{\gamma e}}\left[\omega^2(\tau_{e\gamma} + \tau_{\gamma e}) - \tau_{\gamma e}\omega_r^2 - \tau_{e\gamma}\omega_m^2\right] = 0. \qquad (1)$$

e. Consider the case of tight coupling: all τ's are much smaller than all $1/\omega$'s. Than the second term on the LHS of the above dispersion relation dominates, and one must have

$$\begin{aligned}\omega \simeq \omega_0 &= \sqrt{\frac{\tau_{\gamma e}\omega_r^2 + \tau_{e\gamma}\omega_m^2}{\tau_{\gamma e} + \tau_{e\gamma}}} \\ &= \sqrt{\frac{\rho_r\omega_r^2 + \rho_m\omega_m^2}{\rho_r + \rho_m}},\end{aligned}$$

where I have used $\tau_{\gamma e}\rho_m = \tau_{e\gamma}\rho_r$. Is this in agreement with what you'd expect from (c)? Now, take this to the next order: assume that

$$\omega = \omega_0 + \delta\omega \qquad (2)$$

and show that, to the leading order,

$$\delta\omega = -i\frac{\rho_m\rho_r(\omega_r^2 - \omega_m^2)^2}{2(\rho_m + \rho_r)(\rho_r\omega_r^2 + \rho_m\omega_m^2)(1/\tau_{e\gamma} + 1/\tau_{\gamma e})}. \qquad (3)$$

Simplify using $\rho_m \gg \rho_r$, $\omega_r \gg \omega_m$ at decoupling, and obtain

$$\delta\omega = \omega_0 - (1/2)i\omega_r\tau_{\gamma e}. \qquad (4)$$

Then, using $v_{\rm sr} = c/\sqrt{3}$, obtain Equation (11.6.7) of the book.