# Handout 1. Mathematical Preliminaries: dual spaces and tensors

**Linear or vector spaces.** $L$ is a linear space if
(i) *Addition* of members of $L$ is defined: if $\vec{a}$ and $\vec{b}$ belong to $L$ then $\vec{a}+\vec{b}$ also belongs to $L$.
(ii) *Multiplication* of members of $L$ by (real) numbers is defined: if $\vec{a}$ belongs to $L$ then for any (real) number $\lambda$, vector $\lambda\vec{a}$ also belongs to $L$.
(iii) $L$ contains 0 defined so that $\vec{a}+0=\vec{a}$ for any $a$ in $L$.
(iv) All common algebra rules apply: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$, $\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$, etc.

Linear space $L$ is $n$-dimensional if it contains a linearly independent set of $n$ non-zero vectors $\vec{e}_1, ..., \vec{e}_n$ such that any vector in $L$ can be expressed as a linear combination of $\vec{e}_i$'s:

$$\vec{A}=\Sigma_{i=1}^{n}A^i\vec{e}_i. \tag{1}$$

Here $A^i$ are (real) numbers, and we refer to them as components of vector $\vec{A}$. "Linearly independent" means that none of the $\vec{e}_i$'s can be expressed as a linear combination of the others. The vectors $\{\vec{e}_i\}$ are refered to as the *basis*.

**Einstein summation convention.** This is an extremely convenient shorthand which is used extensively in this course and throughout theoretical physics. It states that whenever a mathematical expression has two indices marked with the same letter, one at the top and the other at the bottom, it is assumed that these indices are summed, without explicitly writing the $\Sigma$ sign. For example, in this convention the vector $\vec{A}$ of equation (1) is written simply as

$$\vec{A}=A^i\vec{e}_i. \tag{2}$$

The repeated indices are refered to as "dummy indices". From now on we shall always use the summation convention.

**Change of basis.** A linear space has infinitely many bases. If $\{\vec{e}_i\}$ form basis of $L$, then vectors $\{\vec{e}'\}$ also form basis of $L$ if (and only if)

$$\vec{e}'_i=\Lambda_i^j\vec{e}_j, \tag{3}$$

where $\Lambda$ is a matrix with non-zero determinant. Here $\Lambda_i^j$ is the element of $\Lambda$ from the $i$'th row and $j$'th column. Note that the summation convention is implied for index $j$.

Equation (3) can be reversed:

$$\vec{e}_i=G_i^j\vec{e}'_j, \tag{4}$$

where the matrix $G$ is the inverse of $\Lambda$, i.e. $G\Lambda=1$. In index notation this can be written as

$$G_i^j\Lambda_j^k=\delta_i^k, \tag{5}$$

where the Kronecker $\delta_i^j$ equals 1 when $i=j$ and 0 otherwise.

The vector $\vec{A}$ is a geometric object and of course does not change when a different basis is chosen. However, its *components* change. In the new basis, they are given by

$$A'^i=G_j^iA^j. \tag{6}$$

Prove it (this is exercise 1 from this week's problem set)!

In General Relativity vectors are also called "contravariant vectors", for reasons which will become clear shortly. Some GR textbooks *define* vector as a basis-dependent set of numbers which transform according to equation (6) when a different basis is chosen. This definition is mathematically correct but hides the geometric nature of vectors.

**One-forms and dual spaces**. General Relativity operates with geometric (i.e., basis-independent) objects. A vector is one example of such object. Another example is a real-valued function $p(\vec{a})$ on a linear space, i.e. a mapping from the linear space onto the real numbers. A *one-form* is a linear function on the linear space, i.e. $p$ is a one-form if

$$p(\alpha\vec{a} + \beta\vec{b}) = \alpha p(\vec{a}) + \beta p(\vec{b}) \tag{7}$$

for any real numbers $\alpha$ and $\beta$ and vectors $\vec{a}$ and $\vec{b}$.

Consider now all of the one-forms on some linear space $L$. Interestingly, the collection of one-forms can be thought of as a linear space. Indeed, the sum of one-forms and multiplication of a one-form by real numbers are trivially defined:
(i) $(p+q)(\vec{a}) = p(\vec{a}) + q(\vec{a})$,
(ii) $(\lambda p)(\vec{a}) = \lambda p(\vec{a})$; here $p$ and $q$ are one-forms and $\lambda$ is a real number. One can check that both $p+q$ and $\lambda p$ defined in this way are one-forms. Moreover,
(iii) a zero one-form $0_{\rm of}$ is defined by

$$0_{\rm of}(\vec{a}) = 0, \tag{8}$$

so $p + 0_{\rm of} = p$ for any one-form $f$. Finally, with these definitions we see that
(iv) all the common rules of algebra apply to one-forms, i.e. $p + q = q + p$, $\lambda(p+q) = \lambda p + \lambda q$, etc.

Lets denote by $L^*$ the linear space of all the one-forms on $L$. It is known as the *dual*, or *conjugate* space to $L$. Although seemingly somewhat artificial, $L^*$ is no less worthy than $L$ itself. This can be seen from a simple but remarkable fact: *If $L$ is n-dimensional than $L^*$ is n-dimensional*
Indeed, let $p$ be any one-form and $\vec{A}$ be any vector:

$$p(\vec{A}) = p(A^i\vec{e}_i) = p(\vec{e}_i)A^i. \tag{9}$$

We see that $p$ is completely determined by $n$ numbers, $\{p(\vec{e}_i)\}$! Thus, $L^*$ is $n$-dimensional.

**Basis in a dual space.** Given a basis $\{\vec{e}_i\}$ in $L$, it is convenient to use a set $\{f^i\}$ of one-forms defined by the following equation:

$$f^i(\vec{e}_j) = \delta^i_j. \tag{10}$$

These one-forms clearly form a basis in $L^*$, called the dual basis, since there are $n$ of the $f^i$s and they are linearly independent (prove it!). Thus, any one-form $P$ can be written as

$$P = P_i f^i, \tag{11}$$

where $P_i$ are real numbers. Furthermore,

$$P(\vec{A}) = P_i A^i. \tag{12}$$

Note that mathematically $P(A)$ is similar to the inner-product of two vectors, $P$ and $A$, except that $P$ and $A$ live in different spaces. We shall return to this analogy below, when we talk about inner products.

When the basis in $L$ changes according to equation (3), the dual basis also changes

$$f'^i = G^i_j f^j; \tag{13}$$

note that this is identical to how *components* of a vector in $L$ change [see eq. (4)]. The one-form components, on the other hand, change in the way identical to that of the basis in $L$:

$$P'_i = \Lambda^j_i P_j. \tag{14}$$

One-forms are also known as covariant vectors or as rank-1 covariant tensors. Some textbooks *define* covariant vectors as a basis-dependent set of numbers which change according to equation (14) when a different basis is chosen (these textbooks do not use the word "one-form"). .

**Dual of dual.** We have defined one-forms as linear real-valued functions on some linear space $L$. But we can also think of vectors as linear real-valued functions on $L^*$! Indeed: let $\vec{a}$ be a vector in $L$. It generates the mapping from $L^*$ to real numbers:

$$f \to f(\vec{a}). \tag{15}$$

This mapping is obviously linear. Therefore, if you live in $L^*$, you will view vectors in $L$ as one-forms. Indeed, for finite-dimensional spaces, it is straighforward to prove that $L$ is the dual space for $L^*$, i.e. that $L^{**} = L$ (this is a good exercise for you).

**Tensors.** So far we have defined two geometric objects: one-form, which is a linear function on $L$, and vectors, which can be thought of as linear functions on $L^*$. But why not consider linear functions of several variables? Indeed we should; these functions are called tensors. To keep things general, we will allow that some of the variables belong to $L$, while others belong to $L^*$. Here is the precise definition of a tensor and the associated terminology:
A tensor of type $(n, m)$ is a real-valued *multilinear* function of $n + m$ variables, where $n$ of the variables live in $L$ and $m$ of them live in $L^*$. By "multilinear" we mean linear with respect to each variable. The integer number $n + m$ is refered to as the rank of a tensor. Clearly, a vector is type (0,1) tensor, and a one-form is type (1,0) tensor.

Let $T$ be a type $(n, m)$ tensor:

$$T = T(x_1, ..., x_n, y_1, ..., y_m) \tag{16}$$

where all $x_i$ belong to $L$ and all $y_j$ belong to $L^*$. Now lets expand all $x_i$ and $y_j$ in the bases of their respective spaces:

$$\begin{aligned} x_i &= {x_i}^l \vec{e}_l \\ y_j &= {y_j}_k f^k. \end{aligned}$$

Substituting these into equation (16) we get

$$T(x_1, ..., x_n, y_1, ..., y_m) = T^{k_1 k_2 ... k_m}_{l_1 l_2 ... l_n} {x_1}^{l_1} {x_2}^{l_2} ... {x_n}^{l_n} y_{1_{k_1}} y_{2_{k_2}} ... y_{m_{k_m}}, \tag{17}$$

where

$$T^{k_1k_2...k_m}_{l_1l_2...l_n} = T(\vec{e}_{l_1}, \vec{e}_{l_2}, ..., \vec{e}_{l_n}, f^{k_1}, f^{k_2}, ..., f^{k_m}). \tag{18}$$

Note that the tensor is completely determined by a set of numbers $T^{k_1k_2...k_m}_{l_1l_2...l_n}$. These numbers are called *components* of tensor $T$. When we change the basis, the tensor remains the same since it is defined in a basis-independent way, but its components change. In the new basis they are given by

$$T'^{i_1i_2...i_m}_{j_1j_2...j_n} = G^{i_1}_{k_1} G^{i_2}_{k_2} ... G^{i_m}_{k_m} \Lambda^{l_1}_{j_1} \Lambda^{l_2}_{j_2} ... \Lambda^{l_n}_{j_n} T^{k_1k_2...k_m}_{l_1l_2...l_n}. \tag{19}$$

Many GR textbooks *define* a tensor as a basis-dependent set of numbers which trasform according to equation (19) when a different basis is chosen.

There are several interesting and useful mathematical operations which can be done with tensors to obtain new tensors, among them *contraction* and *direct product*. You will work through these in your exercises.

**Inner product and metric tensor.** You are all familiar with the inner product: it takes two vectors, $\vec{a}$ and $\vec{b}$, and makes a real number $\vec{a} \cdot \vec{b}$. Recall the properties which define the inner product:
(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
(ii) $(\alpha\vec{a} + \beta\vec{b}) \cdot \vec{c} = \alpha\vec{a} \cdot \vec{c} + \beta\vec{b} \cdot \vec{c}$
Sometimes it is added that $\vec{a} \cdot \vec{a} \geq 0$, but we shall not assume it since in Minkowski space this is not true for space-like vectors. Clearly, the inner product is a rank 2 covariant tensor, known as the *metric tensor*. It is universally denoted by $g$:

$$g(\vec{a}, \vec{b}) = \vec{a} \cdot \vec{b}. \tag{20}$$

It is symmetric, i.e. $g(\vec{a}, \vec{b}) = g(\vec{b}, \vec{a})$. The components of a metric tensor are denoted by $g_{ij}$:

$$g_{ij} = g(\vec{e}_i, \vec{e}_j). \tag{21}$$

Since $g_{ij}$ is a symmetric matrix, a basis can be found in which it takes diagonal form (i.e., only $g_{ii}$ are non-zero). This is the familiar orthogonal basis.

The metric tensor introduces a natural correspondence between vectors in $L$ and one-forms in $L^*$. Let $\vec{a}$ be a vector in $L$. Now consider a one-form $p$ defined by

$$p(\vec{y}) = g(\vec{a}, \vec{y}), \tag{22}$$

where $\vec{y}$ is a variable in $L$. Lets find the components of this one-form:

$$p_i = p(\vec{e}_i) = g_{ij}a^j. \tag{23}$$

It is customary to write the components of the one-form which is connected in this way to vector $\vec{a}$ simply as $a_i$. Thus one writes

$$a_i = g_{ij}a^j. \tag{24}$$

This is called "lowering an index". It is a very convenient notation. For example,

$$\vec{a} \cdot \vec{b} = g_{ij}a^ib^j = a_jb^j = a^ib_i. \tag{25}$$

If the matrix $g_{ij}$ is non-singular (i.e., has an inverse), then from equation (23) we see that the $g$-induced correspondense between vectors and one-forms is one-to-one. This means that for any one-form $p$ one can find vector $\vec{a_p}$ such that

$p(\vec{y}) = g(\vec{a_p}, \vec{y})$. The correspondence allows us to naturally introduce a metric $g^*$ on the dual space $L^*$, in the following way:

$$g^*(p,q) = g(\vec{a_p}, \vec{a_q}). \tag{26}$$

When writing the components $g^{*ij}$ of $g^*$, it is customary to omit the "*". Thus one has

$$g^{ij} = g^*(f^i, f^j). \tag{27}$$

One can show (see exercises) that the matrix $g^{ij}$ is the inverse of the matrix $g_{ij}$, i.e. in the index language

$$g^{ij} g_{jk} = \delta^i_k. \tag{28}$$

Thus,

$$a^i = g^{ij} a_j. \tag{29}$$

**Raising and lowering indices.** We have seen how the metric-induced correspondence between $L$ and $L^*$ allows us to convert vectors into one-forms and back. More generally, this correspondence allows us to change a type $(n, m)$ tensor into a type $(n-1, m+1)$ tensor or into a type $(n+1, m-1)$ tensor. Consider a type $(n, m)$ tensor $T(x_1, ..., x_n, y_1, ..., y_m)$, where $x$'s are in $L$ and $y$'s are in $L$. Define a $(n-1, m+1)$ tensor $S$ by the following manipulation of the $x_n$'th slot:

$$S(x_1, .., x_{n-1}, p, y_1, ..y_m) = T(x_1, .., x_{n-1}, \vec{a}_p, y_1, ..y_m), \tag{30}$$

where $p$ is a one-form in $L^*$ and $\vec{a}_p$ is the corresponding vector in $L$. The components of $S$ are given by

$$S^{kj_1..j_m}_{i_1..i_{n-1}} = g^{k\mathbf{i_n}} T^{j_1..j_m}_{i_1..i_{n-1}\mathbf{i_n}}. \tag{31}$$

This is called raising an index (obviously, we can in this way raise any bottom index of a tensor). It is customary to denote components of the tensor with lowered index by the same letter as the original tensor. Thus,

$$T^{kj_1..j_m}_{i_1..i_{n-1}} = g^{k\mathbf{i_n}} T^{j_1..j_m}_{i_1..i_{n-1}\mathbf{i_n}} \tag{32}$$

Analogously, one can make a $(n+1, m-1)$ tensor out of $T$. We leave the details to the reader. In the component language,

$$T^{j_2...j_m}_{i_1..i_n k} = g_{k\mathbf{j_1}} T^{\mathbf{j_1}..j_m}_{i_1..i_{n-1}i_n} \tag{33}$$

**Summary: key concepts.**

1. Dual spaces, contravariant vectors and covariant vectors (one-forms).
2. Tensors as multilinear functions and as sets of components. Direct product and contraction of tensors (see exercises).
3. Metric tensor, raising and lowering indices.