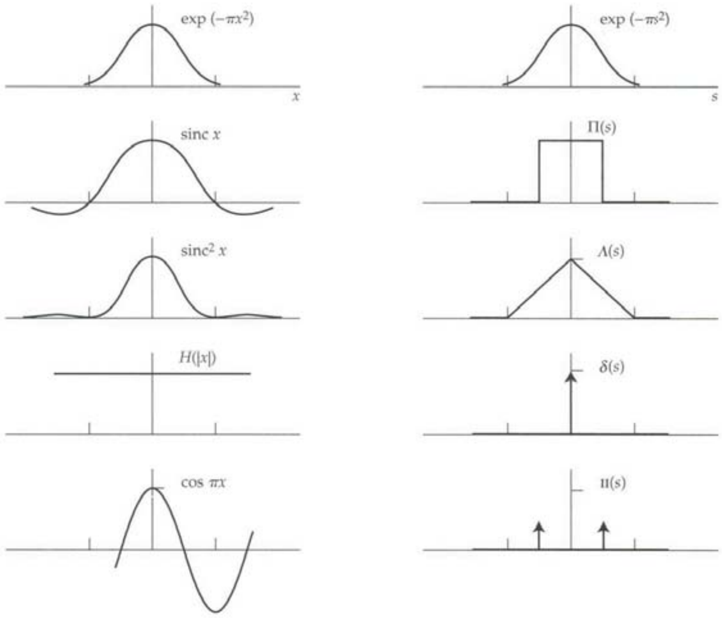
Astronomische Waarneemtechnieken (Astronomical Observing Techniques)

5^{th} Lecture: 1 October 2012



1. Fourier Series
2. Fourier Transform
3. FT Examples in
   - 1D
   - 2D
4. Telescope ⇔ PSF
5. Important Theorems

Sources: Lena book, Bracewell book, Wikipedia

THE FOURIER SERIES

Jean Baptiste Joseph Fourier

From Wikipedia:
Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

A Fourier series decomposes any periodic function or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials).

Application: harmonic analysis of a function *f(x,t)* to study spatial or temporal frequencies.

Fourier Series

Fourier analysis = decomposition using sin() and cos() as basis set.

Consider a periodic function: $f(x) = f(x + 2\pi)$

The Fourier series for *f(x)* is given by:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

with the two Fourier coefficients:

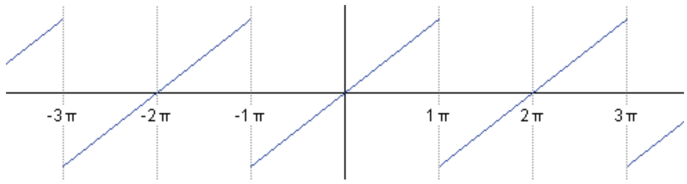
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Example: Sawtooth Function

Consider the sawtooth function:

$$f(x) = x \qquad \text{for} \quad -\pi < x < \pi$$
$$f(x + 2\pi) = f(x)$$



Then the Fourier coefficients are:

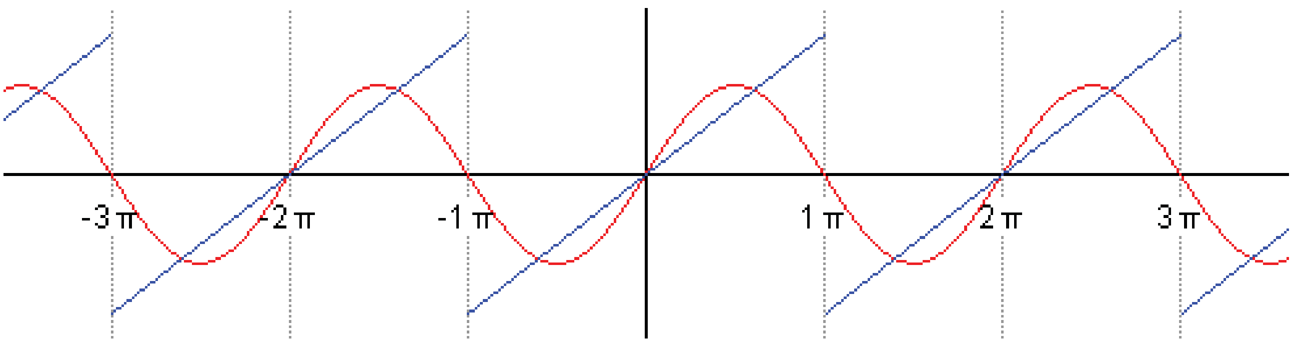
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx \overset{!}{=} 0 \qquad (\cos() \text{ is symmetric around } 0)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = 2 \frac{(-1)^{n+1}}{n}$$

and hence: $$f(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} [\cancel{a_n} \cos(nx) + b_n \sin(nx)] = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

Example: Sawtooth Function (2)

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

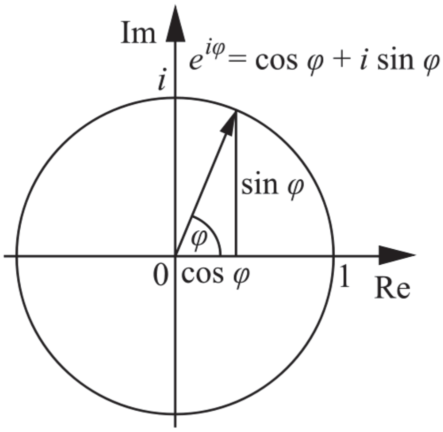


Side note: Euler's Formula

Wikipedia: Leonhard Euler (1707 – 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation.

Euler's formula describes the relationship between the trigonometric functions and the complex exponential function:

$$e^{i2\pi\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$$



With that we can rewrite the Fourier series in terms of the basic waves $e^{i2\pi\theta}$

THE FOURIER TRANSFORM

Definition of the Fourier Transform

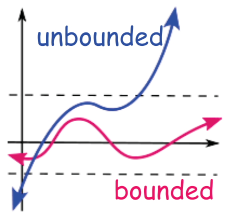
The functions *f(x)* and *F(s)* are called Fourier pairs if:

$$\boxed{F(s)=\int_{-\infty}^{+\infty} f(x)\cdot e^{-i2\pi xs}\,dx}$$

For simplicity we use x but it can be generalized to more dimensions.

The Fourier transform is reciprocal, i.e., the back-transformation is:

$$f(x)=\int_{-\infty}^{+\infty} F(s)\cdot e^{i2\pi xs}\,ds$$



Requirements:

- f(x) is bounded
- f(x) is square-integrable $\int_{-\infty}^{+\infty} |f(x)|^2\,dx$
- f(x) has a finite number of extremas and discontinuities

Note that many mathematical functions (incl. trigonometric functions) are not square integrable, but essentially all physical quantities are.

Properties of the Fourier Transform (1)

SYMMETRY:

The Fourier transform is symmetric:

$$\text{If}\quad f(x)=P_{even}(x)+Q_{odd}(x)$$

$$\Rightarrow F(s)=2\int_{0}^{+\infty} P(x)\cos(2\pi xs)\,dx$$

$$-i2\int_{0}^{+\infty} Q(x)\sin(2\pi xs)\,dx$$

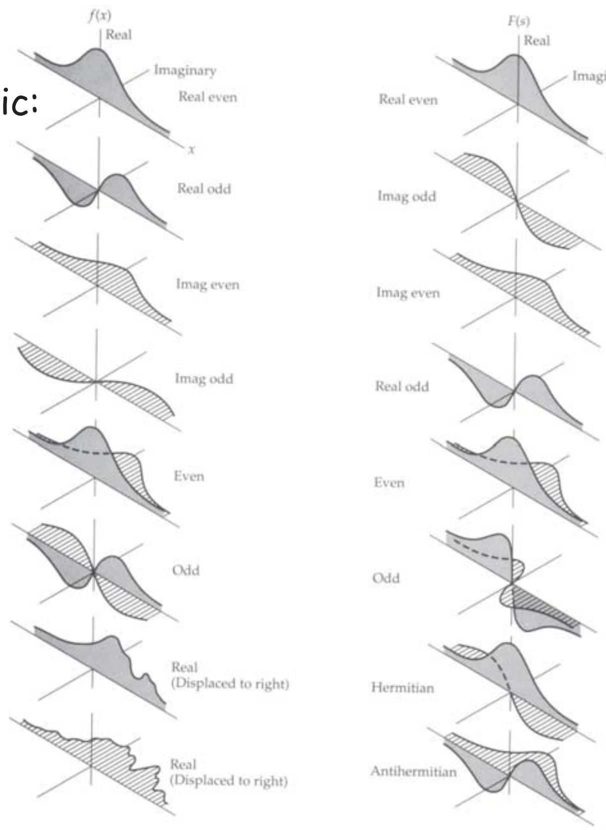


Fig. 2.5 Symmetry properties of a function and its Fourier transform.

Properties of the Fourier Transform (2)

SIMILARITY:
The expansion of a function *f(x)* causes a contraction of its transform *F(s)*:

$$f(x) \rightarrow f(ax) \Leftrightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

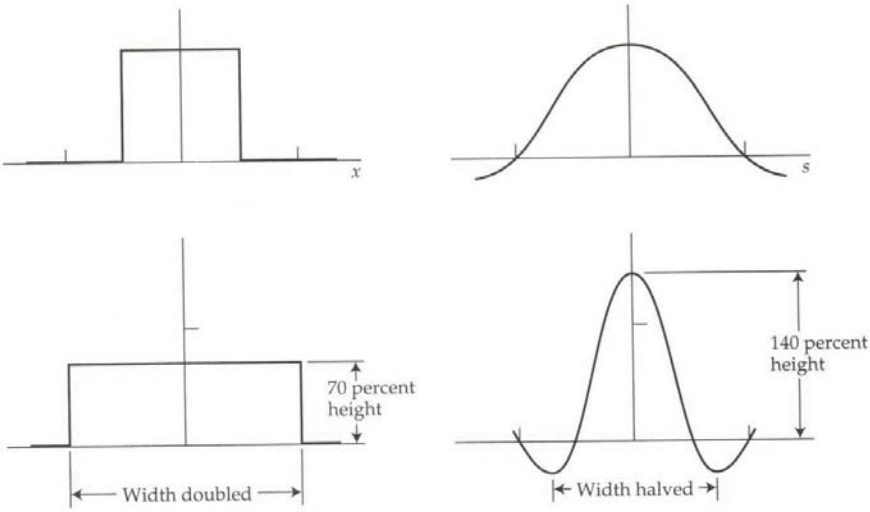


Fig. 6.4 A symmetrical version of the similarity theorem.

Properties of the Fourier Transform (3)

LINEARITY: $F(as) = a \cdot F(s)$

TRANSLATION: $f(x-a) \Leftrightarrow e^{-i2\pi as} F(s)$

DERIVATIVE: $\dfrac{\partial^n f(x)}{\partial x^n} \Leftrightarrow (i2\pi s)^n F(s)$

ADDITION:

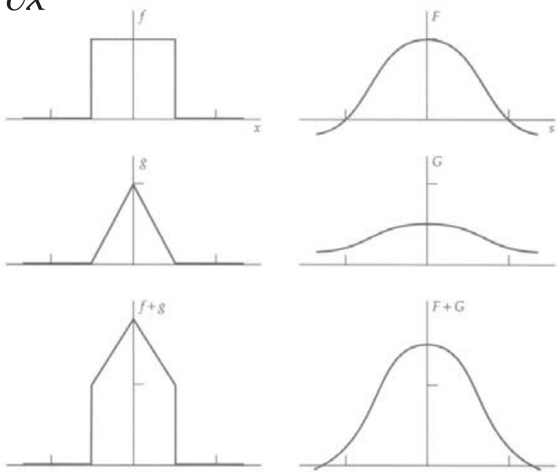
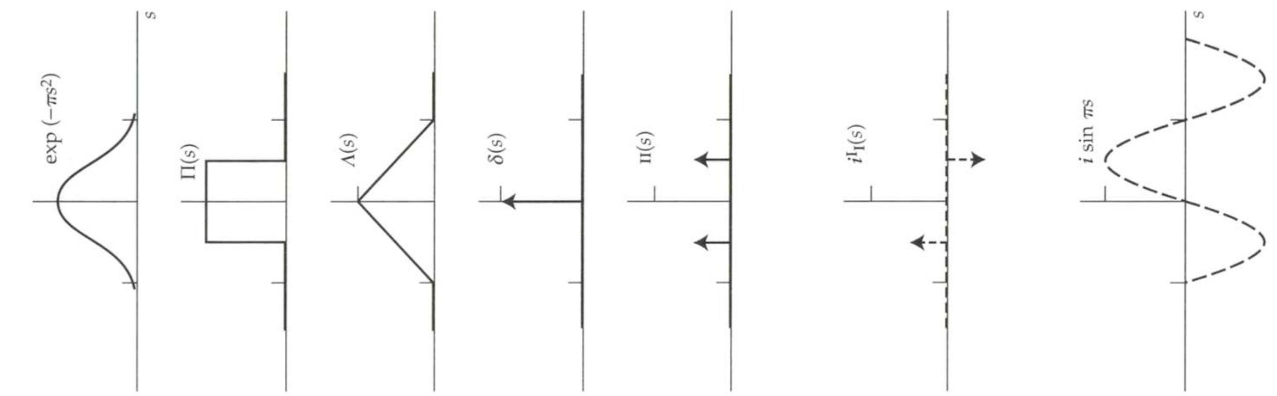
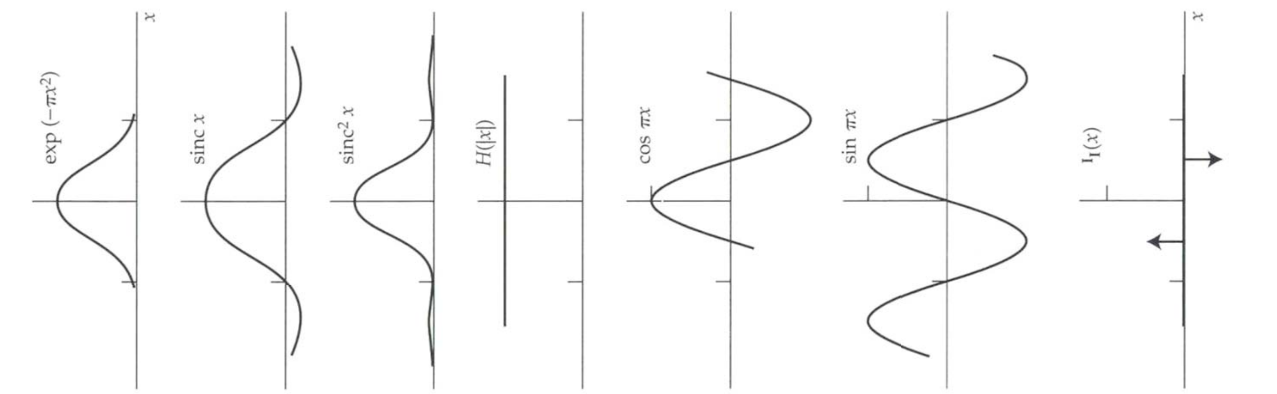


Fig. 6.5 The addition theorem $f + g \supset F + G$.

EXAMPLES OF FOURIER TRANSFORM



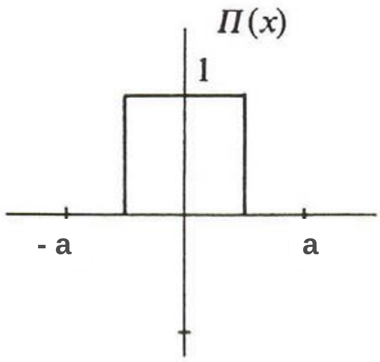
Important 1-D Fourier Pairs



Special 1-D Pairs (1): the Box Function

Consider the box function:

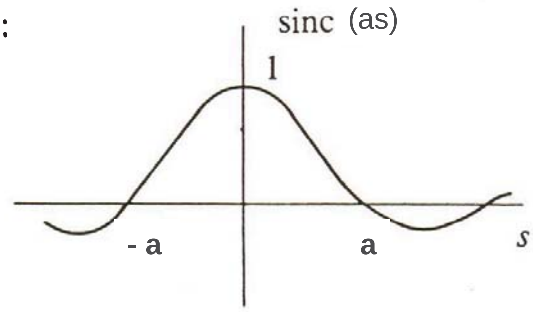
$$\Pi\left(\frac{x}{a}\right)=\begin{cases}1 & \text{for } -\frac{a}{2}<x<\frac{a}{2}\\ 0 & \text{elsewhere}\end{cases}$$



With the Fourier pairs $\Pi(x)\Leftrightarrow\dfrac{\sin(\pi s)}{\pi s}\equiv\operatorname{sinc}(s)$

and using the similarity relation we get:

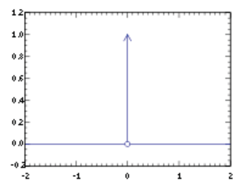
$$\Pi\left(\frac{x}{a}\right)\Leftrightarrow|a|\cdot\operatorname{sinc}(as)$$



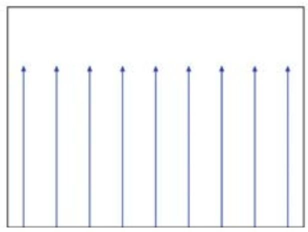
Special 1-D Pairs (2): the Dirac Comb

Consider Dirac's delta "function":

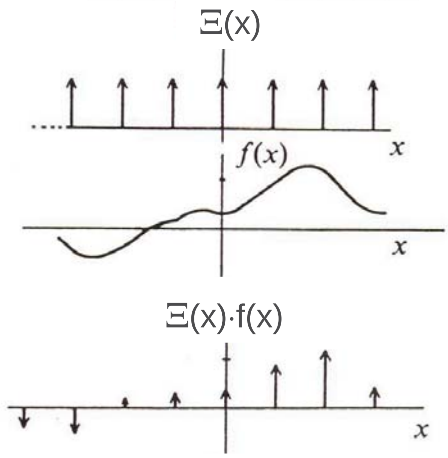
$$f(x)=\delta(x)=\int_{-\infty}^{+\infty}e^{i2\pi sx}dx \quad\rightarrow\quad FT\{\delta(x)\}=1$$



Now construct the "Dirac comb" from an infinite series of delta-functions, spaced at intervals of T:



$$\Xi_{\Delta x}(x)=\sum_{k=-\infty}^{\infty}\delta(x-k\Delta x)\overset{Fourier}{\underset{series}{=}}\frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}e^{i2\pi nx/T}$$



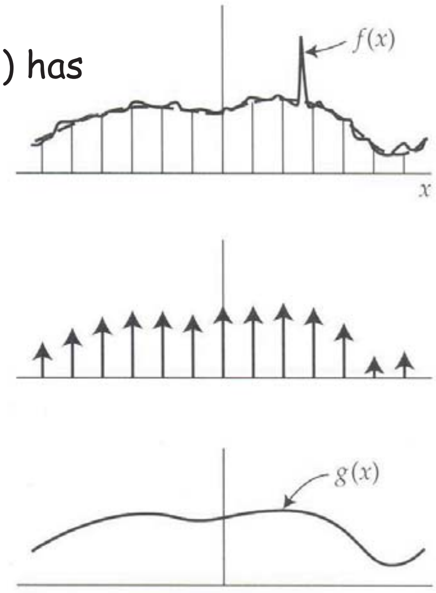
Note:

- the Fourier transform of a Dirac comb is also a Dirac comb
- Because of its shape, the Dirac comb is also called impulse train or sampling function.

Side note: Sampling (1)

Sampling means reading off the value of the signal at discrete values of the variable on the x-axis. $f(x) \rightarrow f(x) \cdot Ш\left(\frac{x}{\Delta x}\right)$

The interval between two successive readings is the sampling rate.
The critical sampling is given by the Nyquist-Shannon theorem:



Consider a function $f(x) \Leftrightarrow F(s)$, where F(s) has bounded support $\left[-s_m, +s_m\right]$.

Then, a sampled distribution of the form

$$g(x) = f(x) \cdot Ш\left(\frac{x}{\Delta x}\right)$$

with a sampling rate of:

$$\Delta x = \frac{1}{2 s_m}$$

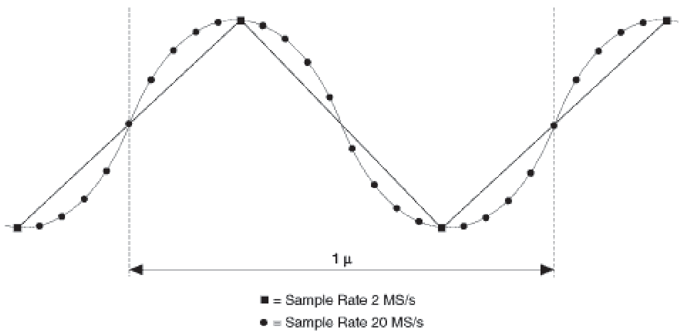
is enough to reconstruct *f(x)* for all *x*.

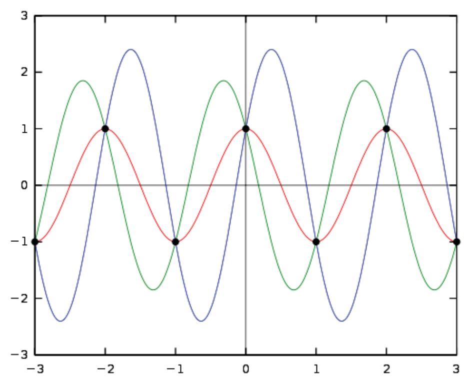
Side note: Sampling (2)

Sampling at any rate above or below the critical sampling is called oversampling or undersampling, respectively.

Oversampling: redundant measurements, often lowering the S/N

Undersampling: measurement dependent on "single pixel"





Aliasing: example: A family of sinusoids at the critical frequency, all having the same sample sequences of alternating +1 and –1. They all are aliases of each other.

Side note: Bessel Functions (1)

Friedrich Wilhelm Bessel (1784 – 1846) was a German mathematician, astronomer, and systematizer of the Bessel functions. "His" functions were first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel.

The Bessel functions are canonical solutions *y(x)* of Bessel's differential equation:

$$x^2 \frac{\partial^2 y}{\partial x^2} + x \frac{\partial y}{\partial x} + \left(x^2 - n^2\right) y = 0$$

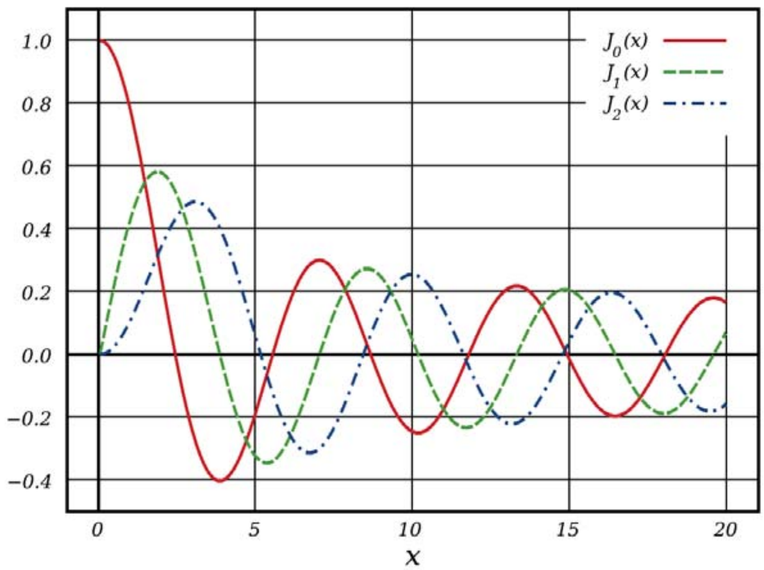
for an arbitrary real or complex number *n*, the so-called order of the Bessel function.

These solutions are: $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}$

Side note: Bessel Functions (2)

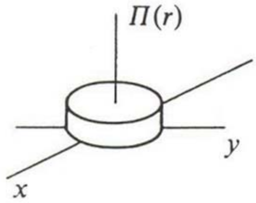
Bessel functions are also known as cylinder functions or cylindrical harmonics because they are found in the solution to Laplace's equation in cylindrical coordinates.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}$$



Special 2-D Pairs (1): the Box Function

Consider the 2-D box function with $r^2 = x^2 + y^2$:

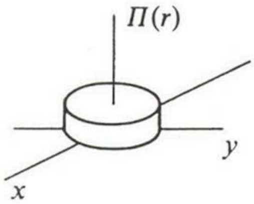


$$\Pi\left(\frac{r}{2}\right) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r \geq 1 \end{cases}$$

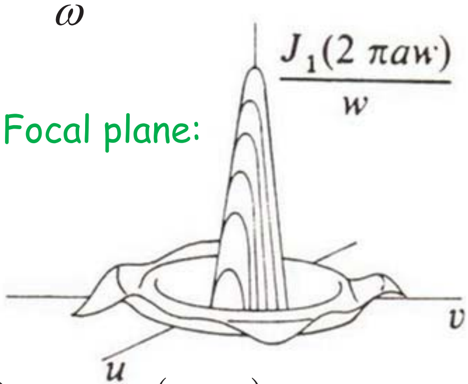
corresponding FT: $\Pi\left(\frac{r}{2}\right) \Leftrightarrow \frac{J_1(2\pi\omega)}{\omega}$ (with 1st order Bessel function J_1)

Example: optical telescope

Aperture (pupil):



Focal plane:

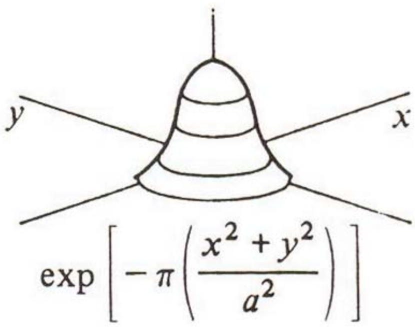


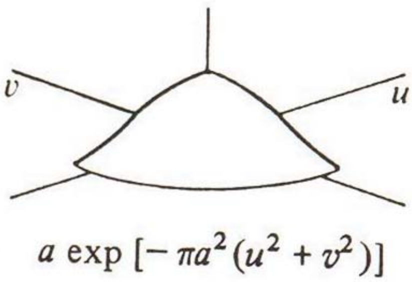
The similarity relation $\Pi\left(\frac{r}{2a}\right) \Leftrightarrow |a| \cdot \frac{J_1(2\pi a\omega)}{\omega}$ means that larger telescopes produce smaller Point Spread Functions (PSFs)!

Special 2-D Pairs (2): the Gauss Function

Consider a 2-D Gauss function with $r^2 = x^2 + y^2$:

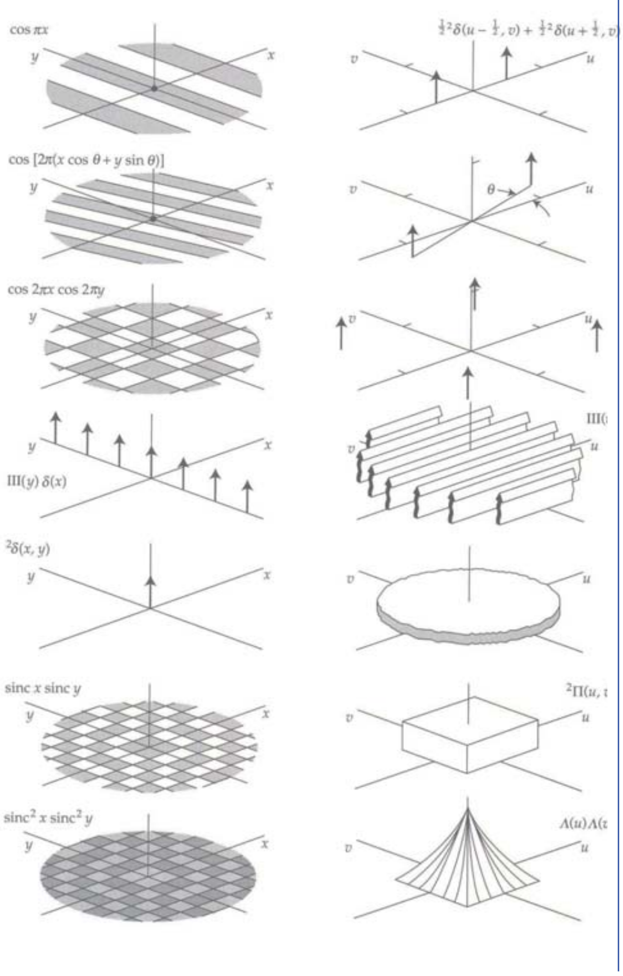
$$e^{-\pi r^2} \Leftrightarrow e^{-\pi\omega^2} \quad \xrightarrow{\text{similarity}} \quad e^{-\pi\left(\frac{r}{a}\right)^2} \Leftrightarrow |a| \cdot e^{-\pi(a\omega)^2}$$

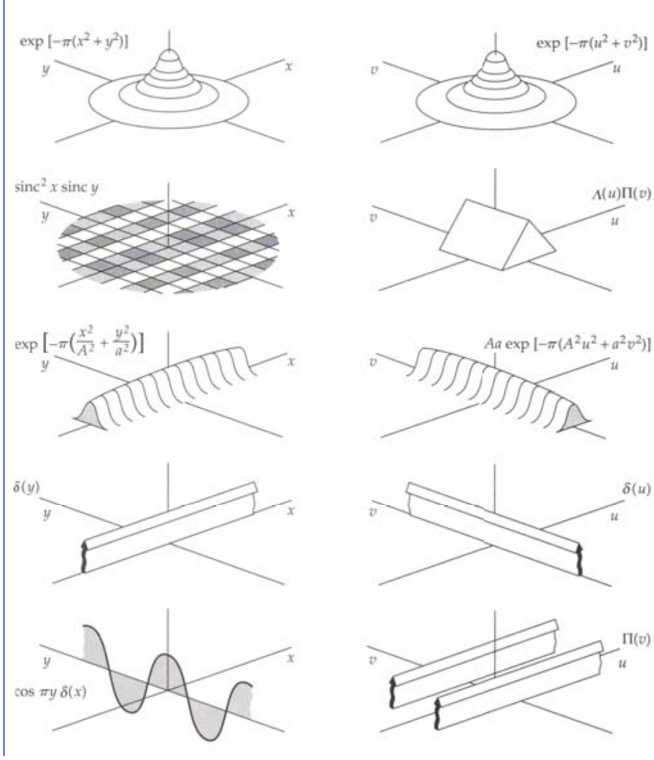




Note: The Gauss function is preserved under Fourier transform!

Important 2-D Fourier Pairs





MORE EXAMPLES:

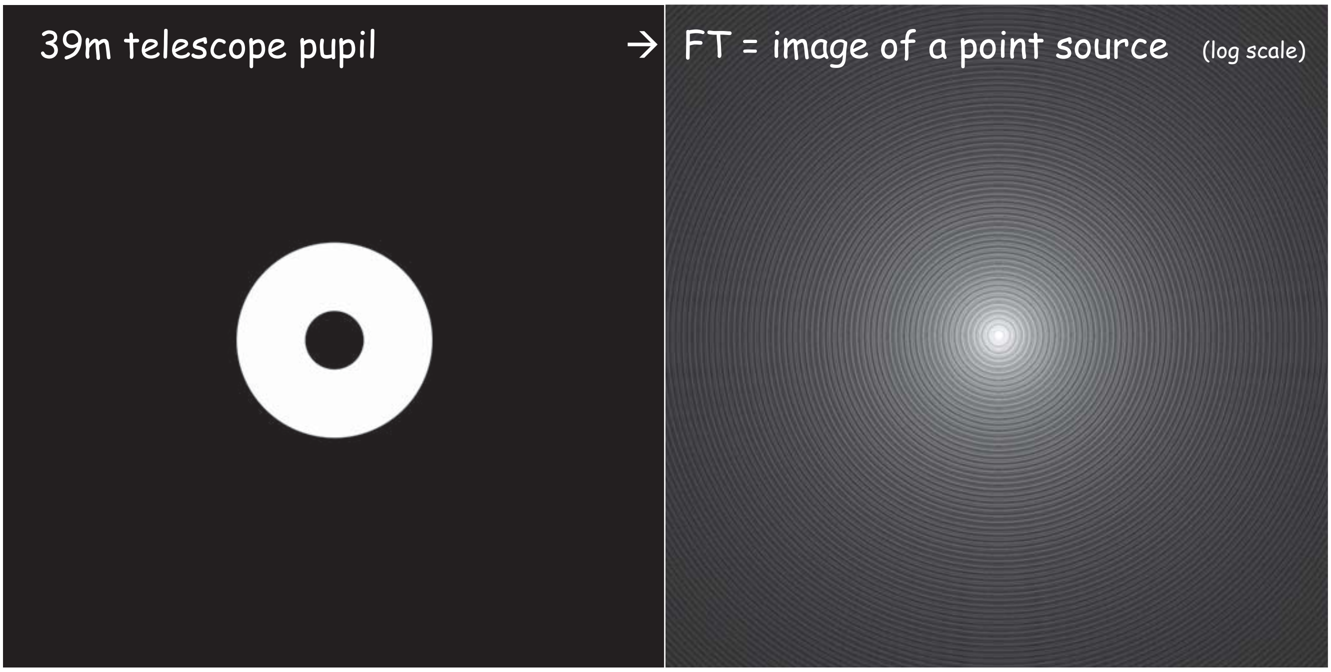
PUPIL (Telescope)

⇔

IMAGE (PSFs)

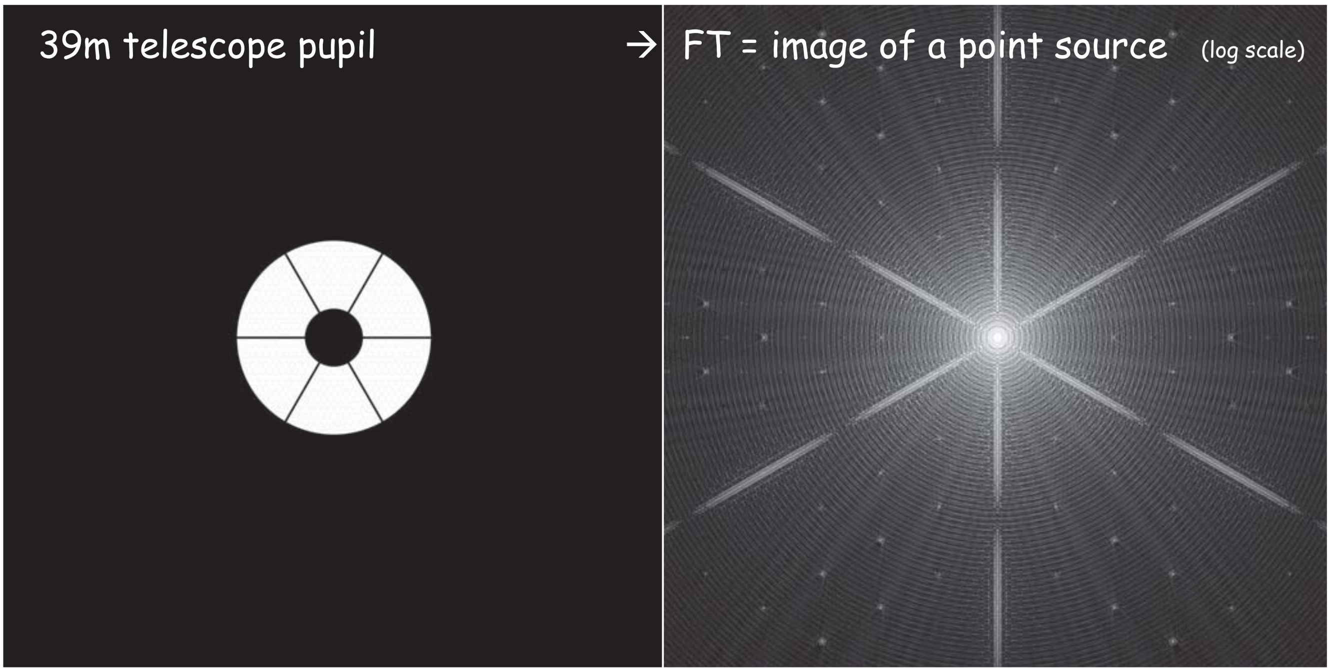
Example 1:

central obscuration,
monolithic mirror (pupil)
no support-spiders



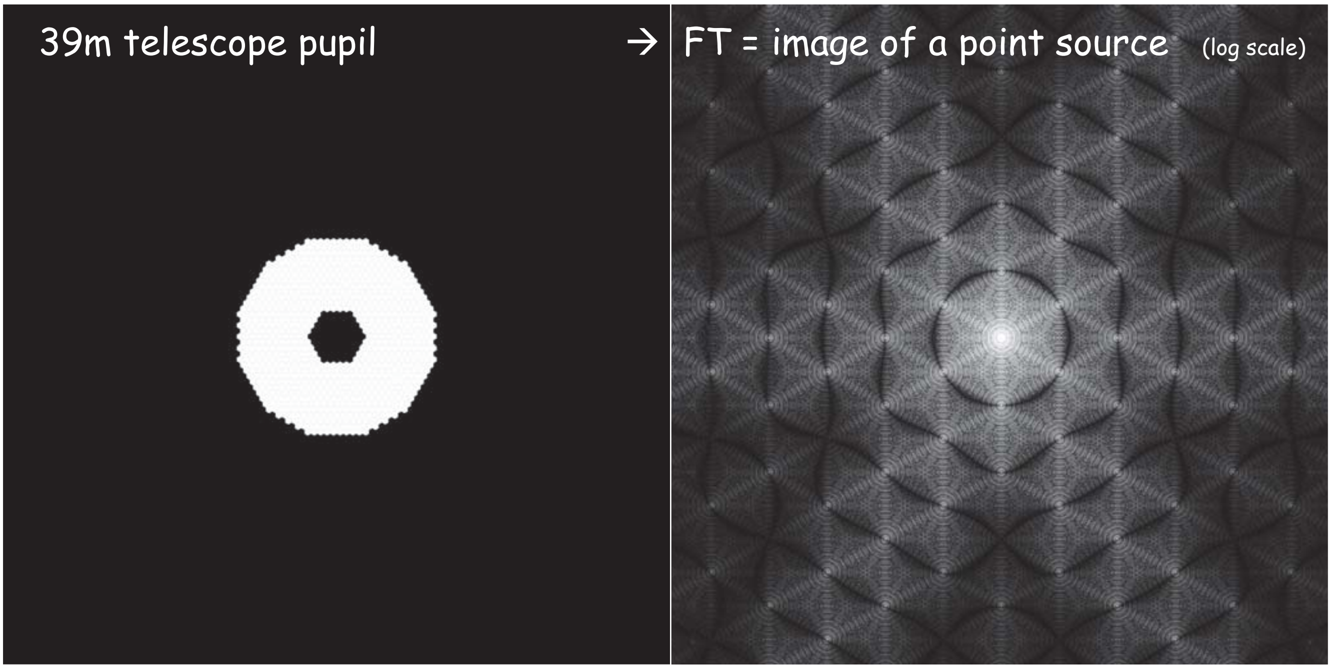
Example 2:

central obscuration,
monolithic mirror (pupil)
with 6 support-spiders



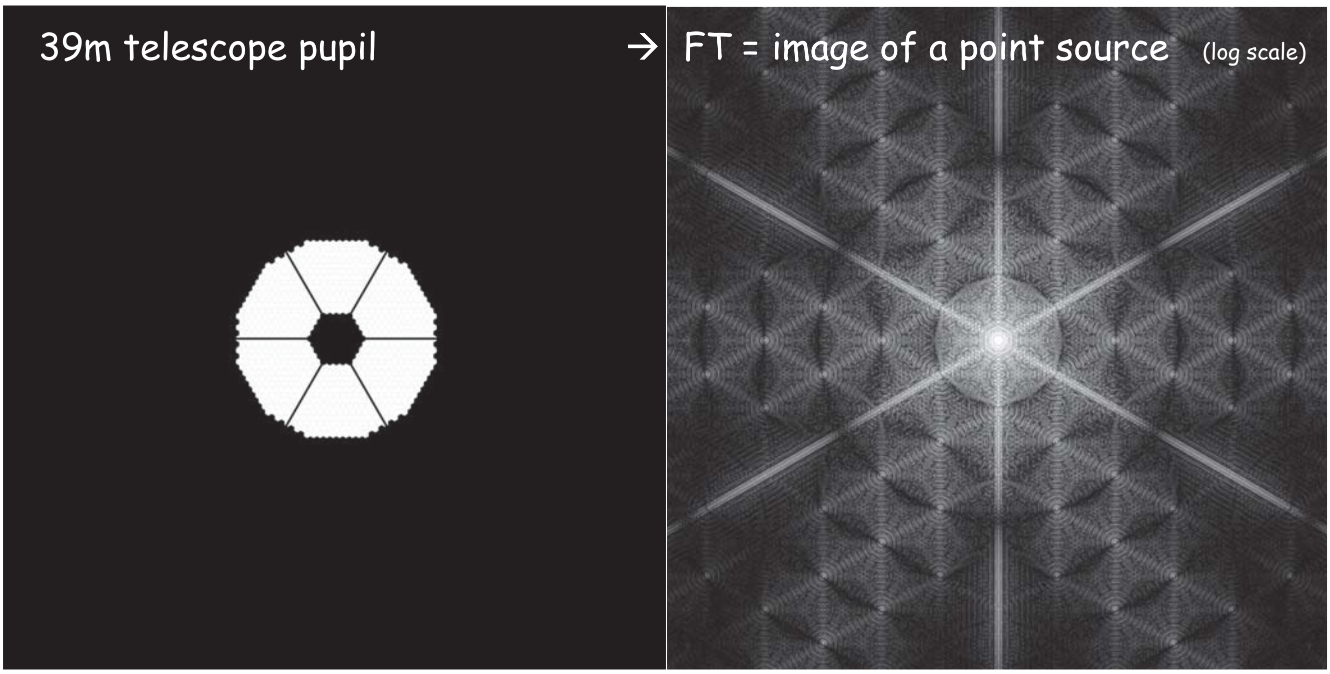
Example 3:

central obscuration,
segmented mirror (pupil)
no support-spiders



Example 4:

central obscuration,
segmented mirror (pupil)
with 6 support-spiders

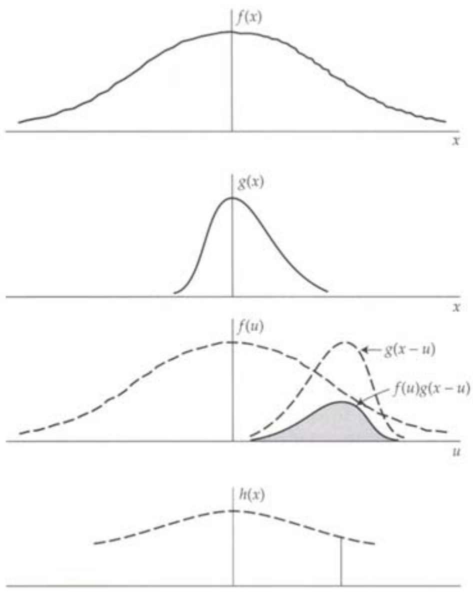


IMPORTANT THEOREMS

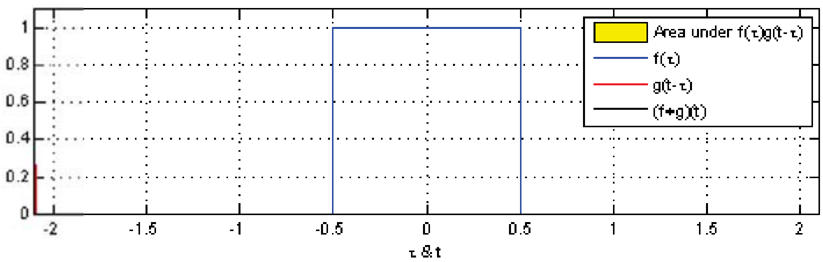
Convolution (1)

The convolution of two functions, $f*g$, is the integral of the product of the two functions after one is reversed and shifted:

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\,du$$



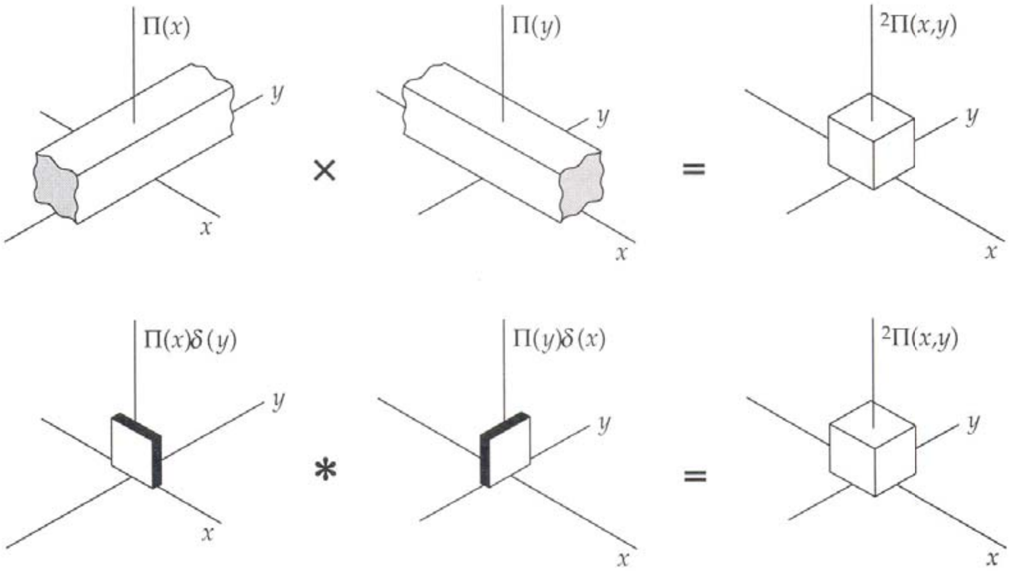
The convolution integral $h(x) = f(x) * g(x)$ represented by a shaded area



Convolution (2)

Note: The convolution of two functions (distributions) is equivalent to the product of their Fourier transforms:

$$\begin{matrix} f(x) \Leftrightarrow F(s) \\ g(x) \Leftrightarrow G(s) \end{matrix} \rightarrow h(x) = f(x) * g(x) \Leftrightarrow F(s) \cdot G(s) = H(s)$$

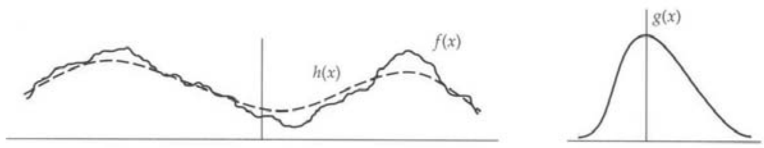


Convolution (3)

Example:
f(x): star
g(x): telescope transfer function
Then *h(x)* is the point spread function (PSF) of the system

$$f(x) * g(x) = h(x)$$

Example:
Convolution of *f(x)* with a smooth kernel g(x) can be used to smoothen *f(x)*



Illustrating the smoothing effect of convolution ($h = f * g$).

Example:
The inverse step (deconvolution) can be used to "disentangle" two components, e.g., removing the spherical aberration of a telescope.

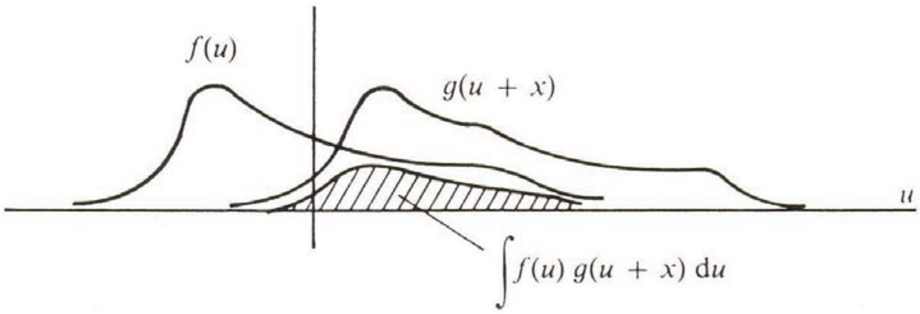
Cross-Correlation

The cross-correlation (or covariance) is a measure of similarity of two waveforms as a function of a time-lag applied to one of them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

The difference between cross-correlation and convolution is:

- Convolution reverses the signal ('-' sign)
- Cross-correlation shifts the signal and multiplies it with another



Interpretation: By how much (x) must *g(u)* be shifted to match *f(u)*? The answer is given by the maximum of *k(x)*

Convolution and Cross-Correlation

The cross-correlation is a measure of similarity of two waveforms as a function of an offset (e.g., a time-lag) between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

Example: search a long duration signal for a shorter, known feature.

The convolution is similar in nature to the cross-correlation but the convolution first reverses the signal ("mirrors the function") prior to calculating the overlap.

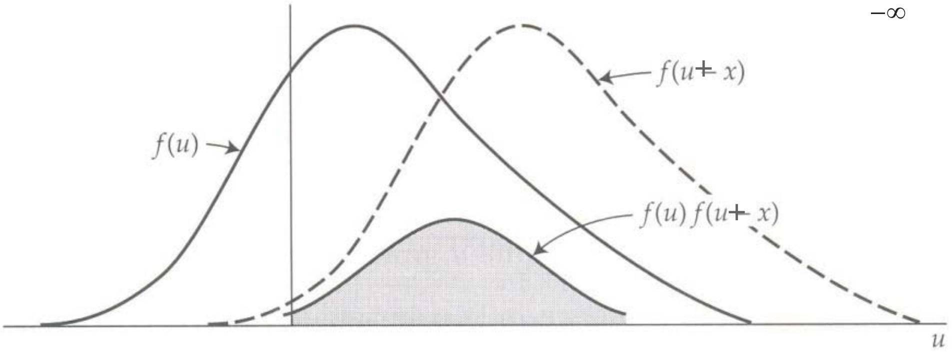
$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$$

Example: the measured signal is the intrinsic signal convolved with the response function

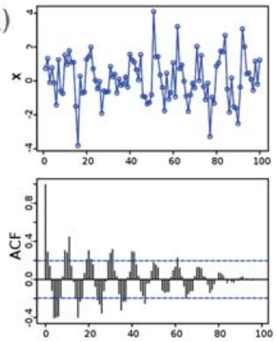
Whereas convolution involves reversing a signal, then shifting it and multiplying by another signal, correlation only involves shifting it and multiplying (no reversing).

Auto-Correlation

The auto-correlation is a cross-correlation of a function with itself:
$$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$$



The autocorrelation function represented by an area (shown shaded)



Wikipedia: *The auto-correlation yields the similarity between observations as a function of the time separation between them.*
It is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal which has been buried under noise.

Power Spectrum

The Power Spectrum S_f of $f(x)$ (or the Power Spectral Density, PSD) describes how the power of a signal is distributed with frequency.

The power is often defined as the squared value of the signal:

$$S_f(s) = |F(s)|^2$$

The power spectrum indicates what frequencies carry most of the energy.

The total energy of a signal is: $\int_{-\infty}^{+\infty} S_f(s)\, ds$

<u>Applications:</u> spectrum analyzers, calorimeters of light sources, …

Parseval's Theorem

Parseval's theorem (or Rayleigh's Energy Theorem) states that the sum of the square of a function is the same as the sum of the square of transform:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(s)|^2 ds$$

Interpretation: The total energy contained in a signal *f(t)*, summed over all times *t* is equal to the total energy of the signal's Fourier transform *F(v)* summed over all frequencies *v*.

Wiener-Khinchin Theorem

The Wiener-Khinchin (also Wiener-Khintchine) theorem states that the power spectral density S_f of a function *f(x)* is the Fourier transform of its auto-correlation function:

$$|F(s)|^2 = FT\{f(x) \otimes f(x)\}$$

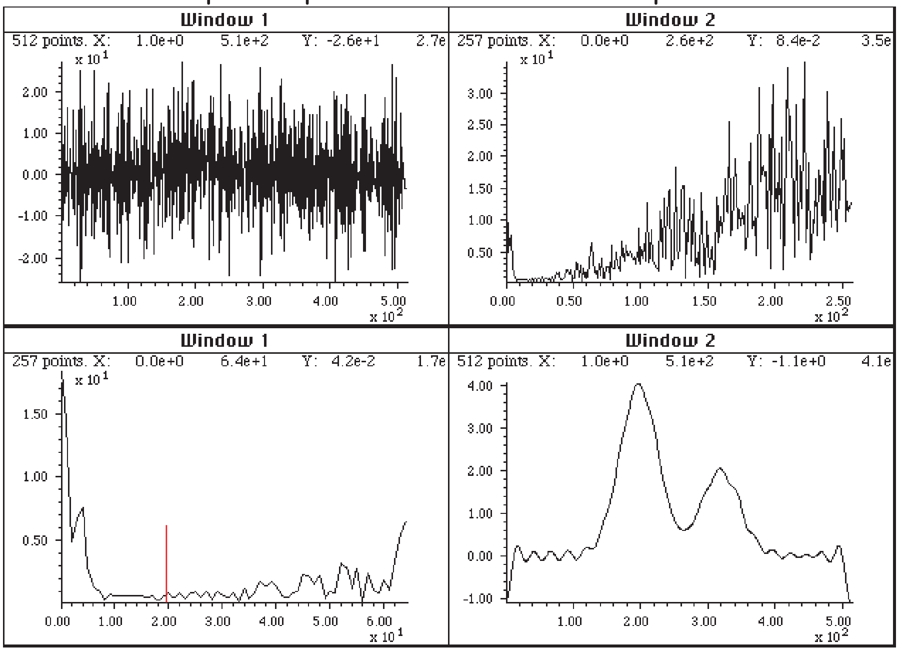
$$\updownarrow$$

$$F(s) \cdot F^*(s)$$

Applications: E.g. in the analysis of linear time-invariant systems, when the inputs and outputs are not square integrable, i.e. their Fourier transforms do not exist.

Fourier Filtering – an Example

Example taken from http://terpconnect.umd.edu/~toh/spectrum/FourierFilter.html



Top left: signal – is I just random noise?
Top right: power spectrum: high-frequency components dominate the signal
Bottom left: power spectrum expanded in X and Y to emphasize the low-frequency region.
Then: use Fourier filter function to delete all harmonics higher than 20
Bottom right: reconstructed signal → signal contains two bands at x=200 and x=300.

Overview

Convolution	$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u)\cdot g(x-u)\,du$
Cross-correlation	$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u)\cdot g(x+u)\,du$
Auto-correlation	$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u)\cdot f(x+u)\,du$
Power spectrum	$S_f(s) = \left\|F(s)\right\|^2$
Parseval's theorem	$\int_{-\infty}^{+\infty} \left\|f(x)\right\|^2 dx = \int_{-\infty}^{+\infty} \left\|F(s)\right\|^2 ds$
Wiener-Khinchin theorem	$\left\|F(s)\right\|^2 = FT\{f(x) \otimes f(x)\}$ $\updownarrow$
	$F(s)\cdot F^*(s)$