Essentials of the analytic model for the expansion of aspherical planetary nebulae

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The hydrodynamics equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} P - \vec{\nabla}\Phi \tag{2}$$

$$\frac{\partial e}{\partial t} + \vec{\nabla} \cdot (\vec{v}(e + P)) = -\rho \vec{v} \cdot \vec{\nabla}\Phi + \Gamma' - \Lambda' \tag{3}$$

with the additional conditions

$$P = \kappa \rho^{\gamma} \tag{4}$$

$$\Delta \Phi = 4\pi G \rho \tag{5}$$

$$k^2 \Phi(k) = 4\pi G \rho(k) \tag{6}$$

$$\Gamma' = \int \Gamma(\rho, T, \cdots)\, d\vec{x} \tag{7}$$

$$\Lambda' = \Lambda(T)\rho^2 \tag{8}$$

are a set of hyperbolic partial differential equations. Therefore, they have weak solutions, i.e. solutions that are only piecewise differentiable. On the interface (discontinuity) between smooth pieces the mass density, momentum density and total energy density are conserved; that is, the flux across the interface is constant. This is expressed in the Rankine-Hugoniot jump conditions connecting the incoming side (0) and the outgoing side (1) of the flow:

$$\rho_0 v_0 = \rho_1 v_1 \tag{9}$$

$$P_0 + \rho_0 v_0^2 = P_1 + \rho_1 v_1^2 \tag{10}$$

$$\epsilon_0 + \frac{P_0}{\rho_0} + \frac{1}{2}v_0^2 = \epsilon_1 + \frac{P_1}{\rho_1} + \frac{1}{2}v_1^2 \tag{11}$$

so that the velocity of the gas ahead of the shock obeys

$$v_0^2 = \frac{(P_1 - P_0)/\rho_0^2}{1/\rho_0 - 1/\rho_1} \tag{12}$$

In a very strong shock, the external pressure P_0 is unimportant, so that we get to good approximation

$$D^2 \equiv v_0^2 = \frac{P_1/\rho_0}{1-\rho_0/\rho_1} \tag{13}$$

In the shock has a complicated shape, the number D is the value of the shock velocity perpendicular to the shock surface. If we then call $f(\vec{x}, t)$ the function that describes the shape of the shock in the course of time, we get

$$D = -\frac{\partial f/\partial t}{|\vec{\nabla} f|} \tag{14}$$

which, by the above argument, can be written as

$$\frac{\partial f/\partial t}{|\vec{\nabla} f|} = -\sqrt{\frac{P_1/\rho_0}{1-\rho_0/\rho_1}} \tag{15}$$

Further manipulation of the jump conditions shows that across a strong shock

$$\frac{\rho_1}{\rho_0} \simeq \frac{\gamma+1}{\gamma-1} \tag{16}$$

and therefore

$$D^2 = \frac{\gamma+1}{2}\frac{P_1}{\rho_0} \tag{17}$$

or, in general,

$$D^2 = \frac{\psi}{\rho_0} \tag{18}$$

This is Kompaneyets's point of departure: he notes that the expression for the shape of the shock can be solved exactly in an exponential atmosphere, i.e. one for which

$$\rho_0 = \rho^* \mathrm{e}^{-z/z_0} \tag{19}$$

If the shock is due to a point explosion, a fixed amount E of energy is injected at an initial time and is subsequenbtly distributed over a volume V. The shock has a closed shape encompassing V, so that

$$P = (\gamma-1)e = \lambda(\gamma-1)\frac{E}{V} \tag{20}$$

The volume enclosed by the shock, which from now on is assumed to be cylindrically symmetric, is

$$V = \pi \int r^2(z,t)\ dz \tag{21}$$

We now substitute Eqs.(17,20,21) into Eq.(14) for the shock normal, use the exponential density distribution, and obtain the partial differential equation

$$\left(\frac{\partial r}{\partial y}\right)^2 - \mathrm{e}^{z/z_0}\left[\left(\frac{\partial r}{\partial z}\right)^2 + 1\right] = 0 \tag{22}$$

where the auxiliary variable y is defined as the time integral of the normal propagation velocity:

$$y \equiv \int_0^t \sqrt{\frac{\lambda E(\gamma^2-1)}{2V\rho^*}}\ dt \tag{23}$$

The miracle is, that this equation is separable. Introducing a separation constant ξ, one obtains

$$r = \xi y + \int_0^z \sqrt{\xi^2 \mathrm{e}^{-z/z_0} - 1}\ dz \tag{24}$$

$$\frac{\partial r}{\partial \xi} = y + \int_0^z \frac{\xi \mathrm{e}^{-z/z_0}}{\sqrt{\xi^2 \mathrm{e}^{-z/z_0} - 1}}\ dz \equiv F(\xi) \tag{25}$$

Because we know that the initial shape of the shock must be spherical, we have $F = 0$, so that ξ can be eliminated by taking the envelope of all the partial solutions to obtain

$$r = 2z_0 \ \arccos\left[\frac{1}{2}\mathrm{e}^{z/2z_0}(1 - (y/2z_0)^2 + \mathrm{e}^{-z/z_0}\right] \tag{26}$$

My method is based on the one found by Kompaneyets, in the sense that it starts from the same strong-shock assumption in a closed expanding volume. There are a few crucial differences, though. First, I assume that the energy is being injected constantly (the interacting-winds hypothesis as generalized by Balick). Second, I do not have an exponential atmosphere, but a slow wind (RGE) expanding with constant velocity from a fixed point.

As before, the propagation of the strong shock that moves into the RGE is governed by one parameter only, namely the ratio of the pressure P_1 behind the shock and the RGE density ρ_0 ahead of it:

$$K \equiv \frac{\gamma+1}{2}\frac{P_1}{\rho_0} \tag{27}$$

Here γ is the adiabatic index of the gas. The quantity P_1 is primarily due to the dynamic pressure of the wind. I adopt spherical coordinates (r, θ, ϕ), but suppress the dependence on ϕ by assuming that ρ_0 is cylindrically symmetric (*this is not necessary for what follows!*). I assume that the RGE gas was deposited by a stationary supersonic wind, so that to a good approximation

$$\rho_0 = \frac{\rho_*}{B(\phi,\theta)}\left(\frac{r_0}{r}\right)^2 \quad \Rightarrow \quad \frac{\rho_*}{A(\theta)}\left(\frac{r_0}{r}\right)^2 \tag{28}$$

If the shape of the shock is described by $r(\theta, t)$, and if λ is the angle between the radius vector to the origin and the normal on the shock surface, then

$$\tan\lambda = -\frac{1}{r}\frac{\partial r}{\partial \theta} \tag{29}$$

and the normal velocity is

$$D = \frac{\partial r}{\partial t}\cos\lambda \tag{30}$$

so that we get the propagation equation

$$D = \frac{\partial r}{\partial t}\left[1+\left(\frac{1}{r}\frac{\partial r}{\partial \theta}\right)^2\right]^{-1/2} \tag{31}$$

Using the strong-shock approximation, we may use $D^2 = \psi/\rho_0$ and therefore

$$\left(\frac{\partial r}{\partial t}\right)^2 = \frac{\psi}{\rho_0}\left[1+\left(\frac{1}{r}\frac{\partial r}{\partial \theta}\right)^2\right]^{-1/2} \tag{32}$$

The function $r(\theta,t)$ describing the shape of the shock front is then a solution of the equation

$$\frac{\partial r}{\partial t} = \sqrt{K + K\left(\frac{1}{r}\frac{\partial r}{\partial \theta}\right)^2} \tag{33}$$

with a known function $K(\theta)$, prescribed by the gas density distribution ahead of the shock, so that

$$\frac{\partial x}{\partial \tau} = \sqrt{A(\theta)\left[1 + (\partial x/\partial\theta)^2\right]} \tag{34}$$

$$x \equiv \log(r/r_0) \tag{35}$$

$$\tau \equiv \frac{t}{r_0}\sqrt{\frac{\gamma+1}{2}\,\frac{P_1}{\rho_*}} \tag{36}$$

Due to the inverse-square behavior of the RGE gas density, we have obtained an equation for the propagation of the shock that is particularly easy to solve: the equation can be decomposed by means of separation of variables. If x is the sum of a function of τ and a function of θ, one readily finds that

$$x = E\,\tau - \int \sqrt{E^2/A(\theta) - 1}\; d\theta + g(E) \tag{37}$$

Here E is an arbitrary constant and $g(E)$ an arbitrary function thereof. From this 'complete integral' we may obtain other solutions of Eq.(34) by taking the envelope of all functions in Eq.(37) for a given range of values of E. If the stellar wind is spherically symmetric x must be constant (which we may take to be zero) at $\tau = 0$ for all values of E; therefore, in our applications we have $g(E) = 0$. Once the RGE density distribution has been specified by means of the function $A(\theta)$, Eq.(37) can be evaluated by quadrature for any desired value of E (Simpson's Rule is accurate enough).

Because it is utterly unknown how the RGE is deposited (see below for references to some possible mechanisms), and *a fortiori* how it acquired its flattening, I had to guess the function A; a plausible expression is

$$A(\theta) = 1 - \alpha + \alpha\exp[\beta\cos(2\theta) - \beta] \tag{38}$$

Here α and β are constants that describe the flattening of the RGE and the steepness of $A(\theta)$. Small values of β represent density distributions of which the cross section contours are ellipses with e/p ratio (equator/pole) $= 1 : \sqrt{1-2\beta}$. Small values of α produce a density that has an e/p ratio close to unity. Large values of β give a density that is nearly independent of θ near the equator, but that drops steeply in a narrow zone near the pole, thus producing an almost spherical RGE cross section with a funnel centered on the symmetry axis.

If one plots the evolution of shock fronts travelling through the density distributions prescribed by Eq.(38), for various values of α and β, it is observed that the shocks all evolve towards a fixed shape in a finite time, typically on the order of $\tau = 4$. This behaviour is generic, and can be shown to be due to the fact that the values of E in Eq.(37) always span a finite range for a bounded function $A(\theta)$.