Reference for astrometry/geodesy:vlbi astrometry/geodesy

Noise in Radio Interferometry

Let's worry a little about the sensitivity of interferometric observations. For the uninitiated we begin with noise and signal strength from radio observations in general. It is rather hard to calibrate the sensitivity of an antenna or an amplifier absolutely, say in millivolts/Jy input, but easier to calibrate the response to a thermal, blackbody source. The output of an antenna from a source, is described in terms of the *antenna temperature* T_A no matter how the signal is amplified or otherwise processed. This is temperature of a hypothetical black-body surrounding the antenna that produces the same signal. The *system temperature* T_S, is the equivalent black-body temperature describing the system output including all sources of noise. So $T_S>0$ even when there is no astronomical source in front of the telescope. Usually $T_S\sim$ physical temperature of the first electronic subsystem, unless the sky and ground around the telescope are effectively hotter. This is the case, for instance, for low frequency observations in the disk of the Galaxy. T_S and T_A can be calibrated by putting resistors with known temperatures in place of the first amplifying stage.

The *brightness temperature* of an extended source is the bb-temperature that produces the same surface brightness at the specified frequency, and is thus a source property, independent of the telescope. From quantum physics:

$$B(T_B,\nu) = \frac{2kT_B}{\lambda^2}\frac{(h\nu / kT_B)}{\exp(h\nu / kT_B) - 1}$$

In units of energy/area/steradian/time/frequency.

Note that the 2nd part of the expression disappears at low frequencies and the expression is linear in T.

For our purposes $T_A \cong T_B \Omega_B / \Omega_A$ where the last ratio is that of the solid angle of the source to that of the antenna beam, or 1 if the source is fully resolved.

The *Flux Density* S_ν of a source is the brightness in physical units integrated over the source, so for low frequencies $S_\nu = \frac{2kT_B}{\lambda^2}\Omega_B$.

For convenience this is often measured in *Janskys* where 1 Jy = 10^-23 erg/s/cm^2/Hz or 10^-26 W/m^2/Hz.

So what is the uncertainty in a measure of *T_A* if the system temperature is *T_B*? From quantum mechanics (to be derived downstream) we can describe a radiation field by the number of modes (polarization, frequency, direction) and the number of photons in each mode. Because photons are *bosons* they do not always obey the usual *sqrt(n)* that we are used to for the standard deviation of a random process. Rather, if the expected number of photons is *n*, the standard deviation of the number is $SD(n) = (n + n^2)^{1/2}$. So this statistical fluctuation is proportional to *n*^1/2 if *n* <<1, but proportional to *n* itself if *n*>>1. This last is true for the radio case. For a rapidly varying field, i.e. one that has a finite bandwidth of $\Delta\nu$ we can measure independent samples of the field at the same rate, and if we do this for a total integration time of *t*, the uncertainty of the averaged power, expressed in terms of T_S is:

$SD(T_A) = T_S\sqrt{t\Delta\nu}$; or signal/noise = $(T_A/T_S)*\sqrt{t\Delta\nu}$.

Homework : A 25m telescope observes an earth-like planet at 21 cm wavelength with a system temperature of 20 K and a bandwidth of 50 MHz for 1 hour. How far away can the planet be so that the signal/noise is >1?

Quantum digression:

In one electronic mode we consider the probable distribution of photon numbers n. From the usual Boltzmann argument, the probability of being in a given state $P(n)$ is proportional to

$$\exp(-E/kT) = \exp(-nh\nu/kT) = \left(\exp(-h\nu/kT\right)^n \equiv Q^n$$

The constant of proportionality can be fixed by insisting that $\sum_n P(n) = 1$. Can you figure this out?

So $P(n) = Q^n * (1-Q)$.To calculate averages, variances etc. consider the m-moments $<n^m>$ where m is an integer

$$<n^m> = \sum_n P(n)n^m = (1-Q)\sum Q^n n^m = (1-Q)(Q\frac{d}{dQ})^m \sum Q^n = (1-Q)(Q\frac{d}{dQ})^m (1-Q)^{-1}$$

This rather complicated looking equation is not so bad for n=0,1,2 So <n> = $\exp(-h\nu/kT)/(1-\exp(-h\nu/kT) = 1/(\exp(h\nu/kT)-1)$.

So the number of photons per mode is large if $h\nu << kT$ and small otherwise.

Combining this with the density of modes, and adding:

- a $h\nu$ to calculate the energy density rather than photon density
- one c to change from energy density to energy flux
- one factor of 2 to account for polarizations:

$$B_\nu(\nu,T) = 2h\nu/\lambda^2/(\exp(h\nu/kT)-1) \cong 2kT/\lambda^2 \; if \;\; h\nu << kT$$

Which is the standard Planck function.

For calculating noise fluctuations it is important to also know <n^2>because we know the uncertainty in a measured value of n is: $\sigma_n^2 = <n^2> - (<n>)^2$

Note for convenience that we can write: $Q = <n>/(1+<n>)$ and $1-Q = 1/(1+<n>)$.

We calculate the formula above for m=2 we find:

$$<n^2> = \frac{Q(1+Q)}{(1+Q)^2} = <n> + 2<n>^2; \sigma_n = \{<n> + <n>^2\}^{1/2}$$

Note that $<n>$ is often written as $\bar{n}$. So for large <n> (radio) the uncertainty is proportional to <n>,while for small <n> (optical) it is proportional to Sqrt(<n>).

Homework: can you show that the above formula for the standard deviation of T_A is correct? (fairly difficult).

What about measuring the phase/strength of an electric wave? Measuring the strength is like measuring then energy; $E = nh\nu$, and measuring the phase is like measuring time as the wave goes by: $t = \phi/\nu$. But we know from the uncertainty principle that $\Delta E \Delta t > h \Rightarrow \Delta n \Delta\phi > 1$

End of quantum digression

So much for 1 telescope. Now 2 telescopes in an interferometer. Let's do this in terms of post-amplification regime where for the measured electric field we have a signal *E* and a noise source $\sigma \boldsymbol{g}$ where *g* is a random gaussian variable with unit variance.

Then the average power for one telescope is

$$< (\boldsymbol{E} + \sigma \boldsymbol{g})^2 > = < \boldsymbol{E}^2 > + 2\sigma < \boldsymbol{Eg} > + \sigma^2 < \boldsymbol{g}^2 > = \boldsymbol{S} + \sigma^2$$

and the Variance of the power is <P^2>-<P>^2 is:

$$< (\boldsymbol{E} + \sigma \boldsymbol{g})^4 > - (\boldsymbol{S} + \sigma^2)^2 = \boldsymbol{S}^2 + 6\sigma^2 \boldsymbol{S} + 3\sigma^4 - [\boldsymbol{S}^2 + 2\sigma^2 \boldsymbol{S} + \sigma^4] = 4\sigma^2 \boldsymbol{S} + 2\sigma^4$$

If the instantaneous S/N is small, the last term dominates and $\sigma_{\boldsymbol{S}} = \sqrt{2}\sigma^2$. If we look at the correlation between two signals this is:

$< (\boldsymbol{E} + \sigma \boldsymbol{g}_1)(\boldsymbol{E} + \sigma \boldsymbol{g}_2) > = \boldsymbol{S}$, where the two gaussian noise variables are independent. Note that the bias term has disappeared because we are using a multiplying correlator; this is very useful.

Easy homework: show that the uncertainty in the interferometric flux is $\sigma_{\boldsymbol{S}\,\mathrm{int}} = \sigma^2$, sqrt(2) better than the single dish (why?).

Several things to contemplate:

- If I have *m* telescopes, I have (m*(m-1))/2 interferometric measurements. If these are statistically independent the uncertainty in the flux of an *unresolved* source goes down as sqrt(2)/m.

- **Easy homework:** show that the signals from two interferometer baselines sharing one telescope are independent.
- Note that expressed in fluxes, the noise from an interferometer is of the same order—perhaps a little better—than the single dish, but expressed in surface brightness sensitivity it is much worse by a factor of $(D/B)^2$, where D is the dish diameter and B the interferometer baseline. This is why, historically, radio interferometers were extremely important in sorting out the structures of synchrotron sources, which are nontherma sources whose brightness temperatures can be $>10^{12}$ K, but observations of thermal sources (HII regions, planets… only became easy when large arrays of telescopes with cryogenic receivers were developed.

Homework: For the VLA at 21 cm wavelength:

- 27 telescopes
- $T_{sys}=20$ K
- Bandwidth 50 MHz

What is the faintest point source (in Janskys) that can be detected in 12 hours?