Fourier Transforms

We have derived the basic relation that the correlation function of the electric field in an aperture is the FT of the brightness distribution on the sky, at least in a small region where a linear approximation is valid. Since FTs appear in this and many other contexts in interferometry, I will spend most of a lesson on their properties.

Definition+notation (one of many)

$FT(f;U) = \int f(x)\exp(iUx)dx$ or

$\widehat{f}(U) = \int f(x)\exp(iUx)dx$

2d: $\hat{f}(\vec{U}) = \int f(\vec{x})\exp(i\vec{U} \bullet \vec{x})d^2x$

inverse: $f(x) = (1/2\pi)\int \widehat{f}(U)\exp(-iUx)dU$

Linear: $FT(af + bg;U) = aFT(f;U) + bFT(g;U)$

Scaling: $\underline{FT(f(ax);) = (1/a)FT(f(x);U/a)}$

Displacement: $FT(f(x-h);U) = \exp(iUh)FT(f(x);U)$

Differentiation: $FT(\frac{df}{dx};U) = -iU * FT(f;U)$

Multiplication*x: $FT(xf;U) = -id(FT(f;U)/dU$

Multiplication *exponential:

$FT(f\exp(iVx);U) = FT(f;U+V)$

Moments: $\mu_n \equiv \int f(x) x^n dx = (-i)^n \frac{d^n}{dU^n} \hat{f}(U)_{U=0}$

Taylor expansion: $\hat{f}(U) = \sum_n \hat{f}^{(n)} U^n / n! = (i)^n \sum_n \mu_n U^n / n!$

Conjugation/reversal/Hermitian : $(FT(f;U))^* = FT(f^*;-U)$

In particular, if f is real $(\hat{f}(U))^* = \hat{f}(-U)$

Special functions: $\delta(x-h) -> \exp(iUh)$

$\exp(-x^2/2\sigma^2) -> \exp(-U^2\sigma^2/2)$

Shah function: (x/a)-> (Ua)…. Square wave, circle…

Special relation with respect to cross correlations:

Up to now we have considered electric fields that were pure sine waves with fixed frequencies. These never occur in nature, even with lasers. All atomic processes emit over finite bandwidths, and we usually receive the radiation from many "atoms", each incoherent with the others. The FT has a special relation with respect to the correlation properties of such incoherent light. Suppose we have a couple of statistical fields measured at two different points, or two different times, and we manage to correlate them: $C = < A(t) B^*(t+h) >$. We assume, for better or worse, that the ensemble average expressed by the <> signs is realized if we take a time average over a long time. We write this in terms of their spectra:

$$C(h) = (1/T)\iiint \hat{A}(\omega)\hat{B}^*(\omega')\exp(it(\omega - \omega'))\exp(i\omega h)d\omega d\omega' dt$$

The integral over t gives a delta function in the frequencies so $C(h) = \int \hat{A}(\omega)\hat{B}^*(\omega)\exp(i\omega h)d\omega$

In particular if A=B, i.e. the same electric field, then the cross correlation is the FT of the spectral power.

FINITE vs. Continuous FTs

All the theorems we have looked at concern continuous functions, but experimentally we usually see data measured at discrete points. We can consider a discrete function as a continuous function multiplied by a **shah** (shin?) function, so the FT is that of the continuous function convolved with a shah function: