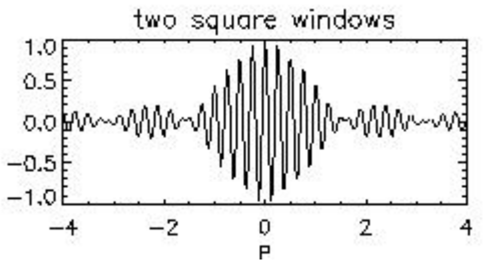
# Chapter 2

Let's go from a lens to a Young style interferometer. Take a square aperture of size X. For now we will only worry about the x-direction. The electric field point spread function in the image plane is: sinc(kX$\alpha$) where $\alpha$=p/f. Let's move it to the left a distance B/2. From simple FT behavior this multiplies the E-field by exp(ikB$\alpha$). This creates no visible change in the image, which is proportional to sinc^2, because we only observe the absolute square of the complex E-field. But now we add a similar square aperture at –B/2. The E-fields add to give 2 cos(kB$\alpha$)sinc(kX$\alpha$)




For the normal case B>>X, you see the very find fringes on top of the larger scale response to the individual holes. This is very characteristic behavior of optical systems, which can be derived from the *convolution theorem* of Fourier Transforms: The FT of a convolution is the product of the FTs of the individual functions, and *visa versa.* So the response of the interferometer to a point source—the so called *point spread function (**psf**)*—is the product of the cosine wave of the interferometer times the psfs of the individual subapertures.

The next step in our construction of an interferometer is to remove the lens. If the holes are small relative to their spacing, the lens-induced phase shift across the hole is small and can be ignored. So we can just ignore the virtually expensive lens in front of the interferometer.

What we have computed so far is the response to an on-axis point source. Suppose the point is off-axis at an angle $\beta$. Then the electric field in the aperture plane will be proportional to exp(ikX β). From another FT theorem, if you shift a function you multiply the FT by an exponential, and again *visa versa*. This is actually a special case of the correlation theorem. So the displacement of the source on the sky yields an exponential phase ramp in the aperture plane, and again a shift in the image plane, thankfully.

## Electric Fields, Power, and Correlation

We don't measure the electric field in the image plane, but its average square, the received power, which in the case of two small holes looks like Sinc^2*cos^2… For more complicated apertures (remember the Besel Functions) we remember from Fourier transformations that the FT of an absolute square is the autocorrelation function of the FT itself. Thus the power psf is the FT of the autocorrelation function of the aperture pattern. The square of a J1(x)/x function is known as an Airy function, and is the FT of the autocorrelation function of a circle.

Normally we cannot measure the electric field directly, because it changes too fast, even at radio wavelengths. We normally measure the power or something related to the power. So let's look at this a little more carefully. We will ignore the single aperture parts of the psf, i.e. consider a very small aperture, and consider only the sinusoidal part at positions *p.* The intensity *I* is the square of the sum of the electric fields from the two pathways.

$$E = E_1 + E_2$$

$$I = E_1^2 + E_2^2 + 2\,\mathrm{Re}(E_1 E_2^*)$$

$$I = E_1^2 + E_2^2 + 2\,E_1 E_2 \cos(\phi)$$

Where f is the phase angle between E1 and E2. The high resolution part of this signal is the last term which oscillates like cos (kBα) while the first two terms represent the total power coming through the 1st and 2nd slits. These may contain very large terms due to sky radiation that have nothing to do with the target, so it would be nice to get rid of them. We can do this if we have a Correlator rather than a detector. A correlator measures the average product of two signals:

$$C \propto \langle E_1 E_2^* \rangle = E^2 \langle \cos(\omega t)\cos(\omega t+\phi) \rangle = E^2 \langle \cos(\phi) - \cos(2\omega t+\phi) \rangle = E^2 \cos(\phi) = E^2 \cos(kB\sin(\alpha))$$

So in the end I can throw away my entire optical system and substitute a *correlator* that takes a piece of the E-field behind each hole and multiplies them to find the *correlated flux: C=Scos(kD)* where *S* is the total flux of the (point) source, and

*D* is the difference in travel distance from the source to the receivers, known as the *Optical Path Difference **(OPD)***.

If $\vec{n}$ is a unit vector toward the source and $\vec{B}$ is the vector connecting the two receivers, then $D = \vec{n} \bullet \vec{B}$. Note that the correlated flux does not depend on where the receivers are placed on the ground, but only on the direction and distance of separation. For *radio* interferometry, the correlation is actually measured directly by digitally multiplying the electronic signals from the two antenna at frequencies up to ~1GHz. I will describe this in more detail in a later lecture. In *optical/infrared* interferometry, where quantum physics plays a role, you cannot do this multiplication directly. Then the E-field product is recovered by splitting the signals, forming the sum and difference, squaring each (i.e. using a power detector), and subtracting: $C = (E_1 + E_2)^2 - (E_1 - E_2)^2 = 4E_1E_2$. To get the total power terms to disappear requires very accurate balancing of the two squaring devices.

## Correlation, coherence, and extended sources

Now let's consider what happens if we have more than one source of radiation on the sky. Then antenna 1 receives not only electric field E1 from one source but also, say, F1 from the other source, with similar E2 and F2 at the other receiver. Then the formulas for I and C should contain complicated terms like:

$$c \approx E^2 \cos(\vec{B} \bullet \vec{n}_E) + F^2 \cos(\vec{B} \bullet \vec{n}_F) + EF(\cos(\vec{B} \bullet \vec{n}_E) + \cos(\vec{B} \bullet \vec{n}_F))$$

That is, besides the sum of the correlated fluxes from each of the sources separately, we get complicated nonlinear cross terms. This would be very messy, but fortunately astronomical sources are incoherent, that is, the phase difference between two unrelated sources is never completely constant, but drifts quickly or slowly with time (to be described later). So a term like $EF\cos(\vec{B} \bullet \vec{n}_E)$ actually shows up as $EF\cos(\vec{B} \bullet \vec{n}_E + \vartheta)$
where φ is a randomly drifting phase difference. Then these terms all average to zero and can be ignored. This would not be true if there were extended coherent sources like *lasers* on the sky.

So, surprisingly or not, the correlated flux from the whole sky, or at least that part of it that our receivers see, is the sum of the fluxes from all the individual parts of the sky; in the end the powers add rather than the electric fields. This can be written symbolically as: $C = \int I(\vec{n})\cos(k\vec{n} \bullet \vec{B})d^2n$
where $I(\vec{n})$ is the intensity as a function of position on the sky. This looks sort of like a Fourier Transform, but not quite. The cosine term is actually a complicated non-linear term of the form cos(kB cos(β)) where β is the angle between $\vec{n}$ and $\vec{B}$. As a function of actual position on the sky, say RA and dec, this is quite complicated. On the other hand, the individual receivers are usually only sensitive to radiation from a small piece of sky, so the equation for the OPD can be *linearized* into:

$$C(U,V) = \int I(\vec{L})\cos(k\vec{U} \bullet \vec{L})d^2L; U = \frac{\partial D}{\partial L}, V = \frac{\partial D}{\partial M}$$

where D is the OPD, and L is the distance relative to a reference position Lo. (U,V) are called the "UV-coordinates" of the particular baseline and reference position.  For measurements with a small bandwidth, like many radio observations, the wavenumber k is absorbed into the definition of U and V, so these are measured in *wavelengths* rather than *meters.*

Ideally, if we were smart enough to design truely *complex* correlators, that measured both the *cosine* and *sine* terms in the integral, we could determine the *complex* correlated flux as: $\hat{C}(U,V)=\int I(L,M)\exp(i\vec{L}\bullet\vec{U})d^2L$ which looks exactly like a Fourier Transform. Can you think of a way of building a complex correlator?

In any case, we have now more or less constructed a complete interferometer.  If we measure the correlated flux at many different telescope separations *U* then we can transform the measured fluxes back into the *L* plane and reconstruct a picture of the source.  This is called *Aperture Synthesis.*  Moving the telescopes around may be difficult or expensive, but part of the problem is solved by keeping them fixed on the ground and letting the world rotate and change the UV coordinates relative to the source.  This is called *Earth Rotation Synthesis.*  A large part of the skill of experienced interferometrists is to reconstruct images when the measurements in the UV plane are not complete.

Exercise: Suppose you have a baseline of length *B* pointing toward a direction on the sky defined by its declination $\delta_B$ and hour Angle:*HA*.  You are observing around a reference point Lo defined by its right ascension and declination, $\alpha_s,\delta_s$.  What are U and V?