Voronoi Vertices as Abell Clusters

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Summary. The large-scale structure of the Universe (around 50 Mpc) can be described by means of "Voronoi foam", a partitioning of space obtained by a process called Voronoi tessellation. We present the first results of a comparison between the statistics of Voronoi foams and the observed galaxy distribution. In particular, we show that the spatial two-point correlation function of Voronoi vertices has a power law behaviour with almost *the same amplitude and slope* as that of the Abell clusters. Moreover, the multiplicity function of the vertices corresponds exactly with that of the clusters, except on the smallest length scales. This confirms our expectations that Voronoi vertices are to be identified with high-density galaxy clusters in the Universe.

1 Introduction

In a pressureless, selfgravitating medium, fluctuations that have a density above average must collapse to form structures with increasingly aspherical shapes, becoming flattened and filamentary (Lynden-Bell 1964; Lin, Mestel & Shu 1965; Icke 1972). The secular increase of the asphericity of perturbations explains the filamentary appearance of the distribution of galaxies (Oort 1983; Giovanelli, Haynes & Chincarini 1986; De Lapparent, Geller & Huchra 1986; Geller 1988), but the approximation breaks down when the perturbations become nonlinear.

Contrariwise, low-density fluctuations expand a little faster than the average Hubble rate, becoming progressively more *spherical* (Icke 1984). These fluctuations are the progenitors of the observed voids in the galaxy distribution (Einasto, Joêveer & Saar 1980; Kirshner *et al.*1981, 1987; for an excellent review, see Rood 1988). Because $| \delta\rho/\rho | < 1$ in a void, the linear approximation remains good for a longer period. The velocity field in the voids is proportional to the distance inside them: voids are "superhubble bubbles."

Thus, we can "turn the Universe inside out", and consider the evolution of the *low*–density regions. We may then think of the large-scale structure of the Universe as a close packing of convex cells out of which matter flows in a slightly super-Hubble expansion.

2 Voronoi foam

The evolution of low-density perturbations (Icke 1984) provides a physical mechanism for the construction of the *skeleton* of the cosmic large-scale mass distribution by tracing the locus of points towards which the matter streams out of the voids. Consider a collection of slightly underdense regions in the primordial density field. These regions are the seeds of voids, expansion centres from which matter flows away until it encounters similar material from an adjacent void. Making the approximation that the excess expansion is the same in all voids, the matter must collect on planes that perpendicularly bisect the axes connecting the expansion centres.

For any given set of expansion centres, or *nuclei*, the arrangement of these planes define a unique process for the partitioning of space, a *Voronoi tessellation* (Voronoi 1908). A particular realisation of this process may be called a *Voronoi foam* (Icke & Van de Weygaert 1987). Each Voronoi cell that surrounds a particular nucleus encloses that part of space which is closer to its nucleus than to any other. In three-space, such a foam is built of three geometrically distinct elements: *walls* that enclose the polyhedral Voronoi voids, *filaments* where three walls intersect, and *vertices* where four filaments come together. The filaments correspond to the elongated superclusters (Icke 1972; Oort 1983; Giovanelli, Haynes & Chincarini 1986), and we identify the vertices with the Abell clusters (Icke & Van de Weygaert 1987). As far as the geometry is concerned, our picture is comparable to that described by Matsuda & Shima (1984) and by Yoshioka & Ikeuchi (1989, preprint), but we emphasise that our model is firmly based on a *physical mechanism*, namely pressure-free selfgravity.

3 Kinematics of Voronoi cell formation

When matter streams out of the voids towards the Voronoi skeleton, cell walls form when material from one void encounters that from an adjacent one. If galaxies form during the expansion of the voids, they can collect in the walls because galaxies have an immense number of internal degrees of freedom, which are excited irreversibly during an encounter (Toomre & Toomre 1972; Binney & Tremaine 1987). Thus, a gas of which the particles are entire galaxies behaves very dissipatively on small length scales.

Accordingly, the kinematics of the formation of Voronoi cells is as follows. Within a void, the mean distance between galaxies increases uniformly in the course of time. When a galaxy tries to enter an adjacent cell, dynamic friction with oncoming galaxies slows its motion; on the average, this amounts to the disappearance of its velocity component perpendicular to the cell wall. Thereafter, the galaxy continues to move within the wall, until it tries to enter the next cell; it then loses its velocity component towards that cell, so that the galaxy continues along a filament. Finally, it comes to rest in a vertex, as soon as it tries to enter a fourth neighbouring void. In a Voronoi foam, there are exactly four cells adjoining each vertex, and the above process is unique.

An immediate consequence of this kinematic behaviour is, that the density in the walls quickly becomes smaller than in the filaments, which, in turn, remain less dense than the vertices, where all matter eventually congregates. This is the main reason why *we identify the vertices with the rich Abell clusters*. The above picture is similar to that of Yoshioka and Ikeuchi (1989, preprint), but more realistic because of the inclusion of kinematic behaviour that is appropriate to dissipative cell formation (Kofman & Shandarin 1988).

We have constructed three-dimensional Voronoi foams geometrically (see Fig. 1; Van de Weygaert 1988, in preparation) and by means of the above kinematic process (Icke 1988, in preparation). An example of a kinematic Voronoi foam, and a slice through it, are shown in Fig.2 . We emphasize that the Voronoi tessellations give a correct asymptotic description of *all* structure formation in gravitating pressureless media, except those in which most of the power in the fluctuations occurs on a scale where dissipation is important (*cf.* Buchert 1988; Gurbatov, Saichev, & Shandarin 1989).

In a box with dimensions $1 \times 1 \times 1$, having periodic boundary conditions, N nuclei were placed at positions $\vec{e}_n$ by a Poisson process (this defines the length scale of the simulation.) Then a Poissonian distribution of G galaxies at positions $\vec{x}_g$ was chosen in the same volume. The motion of each galaxy was determined by a sequence of four trajectories.

First, the nearest nucleus was identified; all distance orderings were obtained quickly by means of a k-d tree sorting method (Van de Weygaert 1987). Let the position of this nucleus be $\vec{e}_1$. The velocity $\vec{u}$ of the galaxy at initial position $\vec{x}_0$ is $\vec{u} = H_*(\vec{x}_0 - \vec{e}_1)$. The excess Hubble parameter was scaled to $H_* = 1$ (this defines the time scale of the simulation.) Each time step Δt, the new position of the galaxy was obtained from the kinematic prescription $\vec{x}_{\rm new} = \vec{x}_{\rm old} + \vec{u}\,\Delta t$. This was repeated until the galaxy got closer to another nucleus (at $\vec{e}_2$, say) than to the one at $\vec{e}_1$. At that instant, the galaxy would move into an adjacent void, but is prevented from doing so by setting the velocity perpendicular to the wall equal to zero, so that the galaxy is constrained to move in the Voronoi wall that bisects $\vec{e}_1 - \vec{e}_2$. The galaxy advances on its new track until yet a third nucleus (at $\vec{e}_3$) comes closer than the previous two.

The component of the velocity perpendicular to the walls that bisect $\vec{e}_1 - \vec{e}_3$ and $\vec{e}_2 - \vec{e}_3$ is now cancelled, so that the galaxy moves along the filament perpendicular to the plane spanned by $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, until finally a fourth nucleus is approached to within the same distance as the first three. At that point, the galaxy has reached a Voronoi vertex: the point where it is equidistant to four nuclei. There, the velocity of the galaxy is finally reduced to zero.

The galactic momentum components perpendicular to walls, to filaments, and in vertices are in reality not exactly cancelled instantly, but they vanish on the average, because the momentum flux on one side of a wall is equal and opposite to the flux on the other side. A residual velocity *dispersion* remains, of course.

The spatial dispersion of the galaxies about the exact Voronoi skeleton is given by the mean value of $|\ \vec{u}\,\Delta t\ |$. This value is on the order of $0.5\,LH_*\,\Delta t$, or about $0.05\,L$ in the simulation shown in Fig.2, where L is the mean radius of a Voronoi cell. In reality, this dispersion is set by the "stickiness" of the interacting galaxies. We have verified that the choice of $L\,\Delta t$ is not important for what follows, provided that $\Delta t \lesssim 0.2$. In Fig.2 the cell structure in the slice is clear, but it is not as crisp in projection. On closer inspection some projected filaments seem to be split; this occurs when one happens to be looking almost along a Voronoi plane.

4 Correlations and multiplicities

We have determined the angular two-point correlation function $w(r)$ of the kinematic simulations (Totsuji & Kihara 1969; Peebles & Hauser 1974; Peebles 1974, 1980). The galaxy positions in the $1 \times 1 \times 1$ cube were projected along the three coordinate directions, and the average of the two-point correlations of the resulting three simulated sky maps was calculated. The results are presented in Fig.3 , showing that $w = 0$ at $r \approx 0.17$. Now $L \approx 0.18$, so that we conclude that the two-point correlation dips through zero at about the Voronoi foam scale, climbing back to zero after that. The observed amplitude of $w(0)$ is on the order of unity (Peebles 1974) which in our kinematic models is reached at the dimensionless time $t \approx 1.8$.

We now compare the spatial correlation function $\xi_{cc}(r)$ of galaxy clusters with the correlation function ξ_{vv} of the Voronoi vertices (Klypin & Kopylov 1983; Shectman 1985; Postman, Geller & Huchra 1986). Our model can be compared with reality because the distances to Abell clusters are accurate enough for reasonable estimates of ξ_{cc} (Bahcall & Soneira 1983; Ling, Frenk & Barrow 1986; see also the review by Bahcall 1988).

Our 3-D Voronoi algorithm was applied to a distribution of 1000 nuclei, giving 6733 vertices in a box with a size of $1 \times 1 \times 1$. We considered Poissonian as well as correlated and anticorrelated distributions of nuclei. ξ_{vv} was determined by using two slightly different estimators. One, used for large radii of the sampling volume, used a Poisson catalog of 25,000 points as a "stochastic reference" (Blanchard and Alimi, 1988, preprint); the other, for small radii, used a more elementary estimator for ξ on those vertices that were further from the box walls than the radius of the sampling sphere.

The resulting cluster-cluster correlation function in our Voronoi model is shown in Fig. 4. The function can be approximated very well by a power law with a slope of 1.97 . In the case of moderately correlated expansion centres, the slope turned out to be 1.88 . Both numbers look quite nice compared with the observationally determined slope of $\gamma = 1.8 \pm 0.3$ (Weinberg, Ostriker, and Dekel, 1988, preprint.)

The amplitude of ξ_{vv} is fixed by $r_0 = 3.0 \times 10^{-2}$; a break appears around $\xi \approx 0.5$, caused by a zero crossing near $r \approx 5.2 \times 10^{-2}$. If we normalize the amplitude of ξ to the mean distance $\bar{n}^{-1/3}$ between clusters, with $\bar{n}$ the mean number density of clusters, we get an estimate for the physical length corresponding to r_0. Abell clusters of richness 1 and above have $r_0 = 55$ Mpc (Blumenthal, Dekel & Primack 1988). This corresponds to $6733^{-1/3} \approx 5.2 \times 10^{-2}$ in the Voronoi model; thus, $r_0 \approx 32$ Mpc, which is the upper limit given in the estimate of r_0 in the review by Bahcall (1988).

The observationally determined ξ_{cc} and our ξ_{vv} differ in the position of the break, which occurs around 54 Mpc in ξ_{vv}, while in the real world the power law extends to 80-90 Mpc (Postman, Geller & Huchra 1986). Possible solutions to this problem are that the nuclei are correlated, or that the Voronoi model and the Abell classification do not attach equal weights to vertices and to clusters.

We think that it is very remarkable that such a simple geometrical picture can give a correlation function which is in accordance with observations, both in slope and in amplitude. It looks as if ξ_{cc} on both small and large scales is determined by the cell structure of the Universe (De Lapparent, Geller & Huchra 1986), which might explain the curious fact that both for $\xi \gg 1$ and for $\xi < 1$ the slope of $\xi(r)$ is the same.

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Figure legenda

Fig.1. Stereoscopic pair of three Voronoi cells sharing a common line. The nuclei are indicated by stars. In a stereo viewer, the dashed lines will appear at the rear of the picture. When attempting stereo fusion with crossed eyes, the pictures must be reversed to obtain the correct depth perception.

Fig.2. Left: kinematic Voronoi foam in a $1 \times 1 \times 1$ cube, projected along one cube axis. The 20 nuclei appear as circles. The 2000 galaxies are indicated as dots; their distribution is Poissonian at time zero (technically, there should be a slight density deficiency near the expansion centres, but this deficit is too slight to be noticed at later times, either visually or in the correlations). The dimensionless time is $t = 1.0$. Right: slice with thickness 0.2 through the cube shown at left (indicated by the arrows).

Fig.3. Angular correlation function of the galaxy distribution of Fig.2. Left: $w(r)$ as determined by counting in linear bins of r. The dashed lines show the $\pm 1\sigma$ deviations as determined from the counting statistics in the three independent coordinate directions. Right: $\log_{10} w(r)$ as determined by counting in logarithmic bins of r. Statistics can be improved easily by running more simulations, but this was not done in order to give a realistic impression of the inherent scatter.

Fig.4. Spatial vertex-vertex correlation functions of a Voronoi foam with 1000 Poissonian nuclei. The left panel shows ξ, on the right is $1+\xi$. Both have been plotted as functions of the dimensionless quantity $x \equiv 2r/\bar{d}$, in which $\bar{d}$ is the mean separation between vertices. The error bars were determined on the basis of $\sqrt{N}$ errors in the estimators. The dashed lines are least-squares fits to all data points except the last three. One finds $\xi(x) = (1.13/x)^{1.97}$ and $1 + \xi(x) = (1.94/x)^{1.77}$. The heavy solid line is an approximate fit to the observational data.